Estimates of Performance Sensitivity of a Stochastic System

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Abstract—Three kinds of estimates of the performance sensitivity of a stochastic system are discussed. The convergence properties of these estimates are investigated. The first estimate using the time average of the derivative of the performance function calculated along a sample trajectory is generally preferable when certain conditions hold for the performance function. The variance of the second estimate using the same input random process is much less than that of the third estimate which uses two different input processes. An example of a one-dimensional linear system with a quadratic performance function is given; it illustrates the general approach to verifying the conditions related to the first estimate for linear systems.

I. INTRODUCTION

CONSIDER a stochastic dynamic system with parameter \( \theta \in R \). The evolution of the dynamic system can be described as a stochastic process \((x(\theta, \omega, t))\), which is a collection of random variables on a sample space \((\Omega, F, P)\) with \( t \in R \) denotes time. For any \( t \) and fixed \( \omega \), \( x(\theta, \omega, t) \) is a function of \( \theta \). Assume that the stochastic process is measurable [14]. Let \( L(x): R \rightarrow R \) be a measurable function. Suppose that the process \((x(\theta, \omega, t))\) is ergodic (hence stationary) and that the performance of the system is defined as the limit of the average value of \( L[x(\theta, \omega, t)] \), i.e.,

\[
J(\theta) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} L[x(\theta, \omega, t)] \, dt.
\]

By ergodicity,

\[
J(\theta) = E\{L[x(\theta, \omega, t)]\}.
\]

In optimization problems, it is important to know the derivative of \( J(\theta) \), which has the following form:

\[
\frac{\partial J(\theta)}{\partial \theta} = \frac{\partial}{\partial \theta} \left( \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} L[x(\theta, \omega, t)] \, dt \right).
\]

(1)

The conventional approaches for estimating \( \partial J/\partial \theta \) are the Monte Carlo methods. Each estimate usually requires two simulations, one with parameter \( \theta \) and the other with \( \theta + \Delta \theta \). However, if the operations in (1) are interchangeable, then we have

\[
\frac{\partial J(\theta)}{\partial \theta} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \frac{\partial L[x(\theta, \omega, t)]}{\partial \theta} \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \frac{\partial L(x)}{\partial x} \frac{\partial x(\theta, \omega, t)}{\partial \theta} \, dt,
\]

with probability 1. (2)

This equation suggests that the time average of \( \partial L/\partial \theta \) can be used as an estimate of \( \partial J/\partial \theta \). If the value \( \partial L/\partial \theta \) can be obtained along a sample trajectory of \((x(\theta, \omega, t))\), this estimate does not require another simulation for the process with parameter \( \theta + \Delta \theta \).

To explore the problem further, we assume that the dynamic equation of the system state is

\[
\frac{dx}{dt} = f(x, \omega, \theta, t).
\]

(3)

Taking the derivative with respect to \( \theta \) on both sides of this equation, we get

\[
\frac{d}{dt} \left( \frac{\partial x}{\partial \theta} \right) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial \theta}.
\]

(4)

This is a linear equation for \( \partial x/\partial \theta \). Thus one can obtain \( \partial J/\partial \theta \) using (2) and (4) instead of (1) and (3). To determine \( \partial x/\partial \theta \) from the linear equation (4) may only require a little extra computation. Therefore, (2) and (4) provide an efficient way of estimating the derivative \( \partial J/\partial \theta \).

In optimization problems, the number of parameters in a system may be large. Using (2) one can obtain estimates of the derivatives of the performance measure \( J(\theta) \) with respect to all parameters at the same time as estimating \( J(\theta) \) by implementing only one simulation of the system and adding some extra computation for solving the linear equation (4). The savings in computation may be remarkable.

Although the estimate using the time average of \( \partial L/\partial \theta \) seems to be superior to crude Monte Carlo methods, the interchangeability of the operators \( \partial / \partial \theta \) and \( \lim_{T \to \infty} \) required by this estimate does not hold for all systems and performance measures. The crude Monte Carlo estimates are necessary in some cases. The objective of this paper is to study the convergence properties of these estimates.

The estimate using the time average of \( \partial L/\partial \theta \) and the two Monte Carlo estimates, one with a common input
random process and the other with two different input processes, are formally defined in Section II. The Monte Carlo estimate using a common input process is a natural extension of the variance reduction technique in simulation using common random numbers [12], [13]. The statistical properties of these three estimates are discussed in Section III for different kinds of stochastic processes. It is shown that the estimate based on (2) converges to $\partial J/\partial \theta$ with probability one if $L(x(\theta, \omega, t))$ is uniformly differentiable with respect to $\theta$ with probability one (w.p.1) on $\Omega$, or the probability that $L(x(\theta, \omega, t))$ jumps in $[\theta, \theta + \Delta \theta]$ is of order $o(\Delta \theta)$. The variances of the two Monte Carlo estimates are found. The convergence properties of these two estimates are studied based on these variances. The variance of the estimate using (2) is usually smaller than those of the crude Monte Carlo estimates. In Section IV, an example is given to illustrate the application of the estimate based on (2). It is proved that the interchangeability of the operators in (2) holds for a one-dimensional linear system with quadratic performance measure. The principle used to prove this interchangeability also applies to multidimensional linear systems.

The one sample trajectory-based estimate was first proposed in [1]. An example in [1] showed that using this estimate one can obtain empirically the optimal value of the Kalman gain of a first-order linear Kalman filter. This paper justifies this estimate by comparing it with other estimates and resolving the problem of interchangeability. The estimate can be used in many engineering problems since systems can often be described by the dynamic equation (3), where $\theta$ can be viewed as a control variable.

Based on the observation made in [1], a new technique known as the perturbation analysis of discrete event dynamic systems was developed in recent years (see, e.g., [2]–[5]). This paper extends some results of perturbation analysis to continuous systems. The convergence properties of estimates based on $N$ independent replications were studied in [7]. Our research enables us to use a single long simulation run instead of $N$ short independent runs to obtain the estimates.

The importance of the one sample trajectory-based estimate is twofold. First, it provides considerable savings in computation for estimating derivatives by simulation. Second, it provides a way of estimating the derivatives of performance of a real system when the system is running. An experiment on the system with a slightly different parameter is not needed. This attractive feature is important for a system manager since changing parameters may not be feasible in a real system.

From the information theoretic point of view, the results indicate that a trajectory of a stochastic system contains not only the information about the system performance, but also that of the sensitivity of the performance. This feature has been studied in many papers for discrete event dynamic systems; one purpose of this paper is to stimulate further research in this area for continuous dynamic systems.

II. Estimates of Performance Sensitivities

We assume that the stochastic process $x(\theta, \omega, t)$ has reached its stationary state and possesses the following ergodic property: for any measurable function $L(x): \mathbb{R} \rightarrow \mathbb{R}$,

$$J(\theta) = E\{L[x(\theta, \omega, t)]\} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T L[x(\theta, \omega, t)] \, dt.$$  

Assume further that the process can be considered as an output of a system (called the shaping filter of the process) whose input is a standard Wiener process, $w(\omega, t)$ (see, e.g., [10] and [11]). If the spectral density function $\Phi(\nu, \theta)$ of $(x(\theta, \omega, t))$ can be factored as

$$\Phi(\nu, \theta) = G(j \nu, \theta)G(-j \nu, \theta),$$

where $j^2 = -1$, then $G(s, \theta)$ is the transfer function of the shaping system. This factorization is possible for all rational functions $\Phi(\nu, \theta)$. In a more general case, the process is governed by an Ito stochastic differential equation [8], [9],

$$dx = f(x, \theta, t) \, dt + g(x, \theta, t) \, dw$$  

where $x$ may be a vector. We assume that $f(x, \theta, t)$ and $g(x, \theta, t)$ satisfy conditions specified by the existence and uniqueness theorems (see [16]) so that (5) has a unique solution.

Let $\chi(\omega, t) = \{w(\omega, s); s \leq t\}$. Then $x(\theta, \omega, t)$ can be expressed as $x[\theta, \chi(\omega, t)]$, and

$$J(\theta) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T L[x[\theta, \chi(\omega, t)]] \, dt.$$  

When no confusion can arise, we simply write $\chi$ for $\chi(\omega, t)$.

By (2) we define the following estimate (estimate 1) of $\partial J(\theta)/\partial \theta$,

$$\text{est.1: } \left[ \frac{\partial L}{\partial \theta} \right]_{T, 1} = \frac{1}{T} \int_0^T \frac{\partial}{\partial \theta} L[x(\theta, \chi(\omega, t))] \, dt.$$  

If the desired interchangeability holds, then

$$\frac{\partial J(\theta)}{\partial \theta} = \lim_{T \rightarrow \infty} \left[ \frac{\partial L}{\partial \theta} \right]_{T, 1}, \text{ w.p.1.}$$  

For any realization of $\chi(\omega, t), L[x(\theta, \chi(\omega, t))]$ is a function of $\theta$, which is called a sample performance function. $\partial L[x(\theta, \chi(\omega, t))]/\partial \theta$ is called the sample derivative of the performance function.

To estimate $\partial J/\partial \theta$ using the crude Monte Carlo methods, two simulations, one for the system with parameter $\theta$ and the other with parameter $\theta + \Delta \theta$, are required. In these two simulations $x(\theta, \omega, t)$ can be stimulated by either the same input process $\chi$ or two different input processes $\chi_1$ and $\chi_2$. This leads to two different estimates of $\partial J/\partial \theta$, which will be called estimates 2 and 3 and denoted by...
More precisely, est. 2:
\[
\frac{\Delta J}{\Delta \theta} \rightarrow_{T, 2} = \frac{1}{T} \left\{ \int_{0}^{T} L[x(\theta + \Delta \theta, x_{1})] \, dt - \frac{1}{T} \int_{0}^{T} L[x(\theta, x_{1})] \, dt \right\}
\]
and
\[
\frac{\Delta J}{\Delta \theta} \rightarrow_{T, 3} = \frac{1}{T} \left\{ \int_{0}^{T} L[x(\theta + \Delta \theta, x_{2})] \, dt - \frac{1}{T} \int_{0}^{T} L[x(\theta, x_{1})] \, dt \right\}.
\]

Estimate 2 is essentially similar to the estimate of the difference of two random variables using common random numbers [12], [13]. From ergodicity and the definition of the partial derivative, we have
\[
\frac{\partial J(\theta)}{\partial \theta} = \lim_{\Delta \theta \to 0} \lim_{T \to \infty} \left\{ \frac{\Delta J}{\Delta \theta} \right\}_{T, 2}, \quad \text{w.p.1,} \tag{7}
\]
and
\[
\frac{\partial J(\theta)}{\partial \theta} = \lim_{\Delta \theta \to 0} \lim_{T \to \infty} \left\{ \frac{\Delta J}{\Delta \theta} \right\}_{T, 3}, \quad \text{w.p.1.} \tag{8}
\]

However, implementing these two double limits in practice is not feasible. One cannot first let $T$ go to infinity for a fixed $\Delta \theta$ and then let $\Delta \theta$ go to zero. $T$ and $\Delta \theta$ must always be positive finite numbers. What one can do is keep $\Delta \theta$ sufficiently small and $T$ sufficiently large. This means that we can only approach $T \to \infty$ and $\Delta \theta \to 0$ along a path $R(\Delta \theta, T)$ in the space $(\Delta \theta, T)$. Therefore, we have to consider the following limits:
\[
\frac{\partial J(\theta)}{\partial \theta} = \lim_{R \to (0, \infty)} \left\{ \frac{\Delta J}{\Delta \theta} \right\}_{T, 2}, \tag{9}
\]
and
\[
\frac{\partial J(\theta)}{\partial \theta} = \lim_{R \to (0, \infty)} \left\{ \frac{\Delta J}{\Delta \theta} \right\}_{T, 3}. \tag{10}
\]

where $R \to (0, \infty)$ means that $\Delta \theta \to 0$ and $T \to \infty$ along a particular path $R$. Note that in (9) and (10) the “limit” may be in different probabilistic senses (e.g., convergence with probability one, in probability, or weak convergence, etc., see [14]). We will specify the meaning of these convergences later for different problems.

We have defined three estimates of $\partial J/\partial \theta$. The fact that (7) and (8) hold does not guarantee that (6), (9), and (10) hold. The statistical properties of these estimates are studied in the next section. The convergence properties of these estimates depend strongly on the properties of the sample performance function $L[x(\theta, \chi(\omega, t))]$. Fig. 1 illustrates the generation of these three estimates. Note that for nonlinear systems, solving the linear equation (4) is usually much simpler than simulating the system with $\theta + \Delta \theta$.

III. CONVERGENCE OF SENSITIVITY ESTIMATES

A. Convergence of Estimate 1

First, assume that $\partial L[x(\theta, \chi(\omega, t))] / \partial \theta$ exists for all $t \in [0, \infty)$ w.p.1, and that the process $\partial L[x(\theta, \chi(\omega, t))] / \partial \theta$ is measurable and ergodic. Equation (6) can be written as
\[
E\left\{ \frac{\partial}{\partial \theta} \left[ EL[x(\theta, \chi(\omega, t))] \right] \right\} = E\left\{ \frac{\partial}{\partial \theta} L[x(\theta, \chi(\omega, t))] \right\}. \tag{11}
\]

Therefore, estimate 1 converges to $\partial J(\theta) / \partial \theta$ w.p.1, as $T \to \infty$ if and only if (11) holds, i.e., the two operators $\partial / \partial \theta$ and $E$ are interchangeable for $L[x(\theta, \chi(\omega, t))]$. The conditions for (11) using $N$ independent replications were derived in [7]. Here we will obtain similar results based on ergodic theory.

As a function of $\theta$, $L[x(\theta, \chi(\omega, t))]$ can be expanded for any $t \in [0, \infty)$,
\[
L[x(\theta + \Delta \theta, \chi(\omega, t))] - L[x(\theta, \chi(\omega, t))]
= \frac{\partial L}{\partial \theta} [x(\theta, \chi(\omega, t))] \Delta \theta + r[\Delta \theta, \chi(\omega, t)] \quad \text{w.p.1} \tag{12}
\]
where
\[
\lim_{\Delta \theta \to 0} r[\Delta \theta, \chi(\omega, t)] = 0 \quad \text{w.p.1.}
\]
Thus
\[
\frac{\partial J(\theta)}{\partial \theta} = \lim_{\Delta \theta \to 0} \lim_{T \to \infty} \frac{1}{T} \int_{T_0}^{T} \left( \frac{\partial L[x(\theta, \chi, \omega, t)]}{\partial \theta} + \frac{r[\Delta \theta, x(\omega, t)]}{\Delta \theta} \right) dt
\]
\[
+ \lim_{\Delta \theta \to 0} \lim_{T \to \infty} \frac{1}{T} \int_{T_0}^{T} \left( \frac{\partial L[x(\theta, \chi, \omega, t)]}{\partial \theta} + \frac{r[\Delta \theta, x(\omega, t)]}{\Delta \theta} \right) dt
\]
\[
= E\left( \frac{\partial L[x(\theta, \chi, \omega, t)]}{\partial \theta} \right) + \lim_{\Delta \theta \to 0} E\left( \frac{r[\Delta \theta, x(\omega, t)]}{\Delta \theta} \right).
\]
Therefore, (11) holds if and only if
\[
\lim_{\Delta \theta \to 0} E\left( \frac{r[\Delta \theta, x(\omega, t)]}{\Delta \theta} \right) = 0.
\] (14)

Other sufficient and necessary conditions exist for the interchangeability in (11). For example, if \( L[x(\theta, \chi, \omega, t)] > 0 \), then interchangeability holds if and only if \( L[x(\theta, \chi, \omega, t)] \) is uniformly integrable [14]. There are also sufficient conditions for (11). The most common one is Lebesgue's dominated convergence theorem. However, it is usually difficult to check these conditions in practice. In the following, we will find other sufficient conditions for (11). These conditions have some practical implications, and can be applied to estimate 1 as well as estimate 2.

**Definition:** \( L[x(\theta, \chi, \omega, t)] \) is said to be uniformly differentiable at \( \theta \) w.p.1 on \( \Omega \), if for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \(|\Delta \theta| < \delta, \) then
\[
\left| \frac{r[\Delta \theta, x(\omega, t)]}{\Delta \theta} \right| < \epsilon, \quad \text{w.p.1.}
\]

By the stationarity of the process, the definition applies to all \( t \in [0, \infty) \). It is easy to see that if \( L[x(\theta, \chi, \omega, t)] \) is uniformly differentiable at \( \theta \), then (14) holds. The proof is straightforward and is omitted.

Next, we consider functions that are discontinuous with respect to \( \theta \). Suppose that \( L[x(\theta, \chi, \omega, t)] \) has the following form,
\[
L[x(\theta, \chi, \omega, t)] = L_1[x(\theta, \chi, \omega, t)] + L_2[x(\theta, \chi, \omega, t)]
\] (15)
where \( L_1[x(\theta, \chi, \omega, t)] \) is uniformly differentiable at \( \theta \), and \( L_2[x(\theta, \chi, \omega, t)] \) is a right continuous, piecewise constant function of \( \theta \) for any \( t \) and any realization of \( \chi \). Note that if \( L_2(x) \) is a piecewise constant function of \( x \), then it is also a piecewise constant function of \( \theta \). An example of this kind of function is \( L_2(x) = 1, \) if \( x > 1 \) and 0, if \( x \leq 1 \). We assume that for any \( \Delta \theta, \) the probability that \( L_2[x(\theta, \chi)] \) (or, equivalently, \( L[x(\theta, \chi)] \)) jumps in interval \([\theta, \theta + \Delta \theta] \) is positive and that the probability that it jumps at \( \theta \) is zero. Then we have
\[
\frac{\partial}{\partial \theta} L_2[x(\theta, \chi)] = 0, \quad \text{w.p.1 at any } \theta
\]
or
\[
\frac{\partial}{\partial \theta} L[x(\theta, \chi)] = \frac{\partial}{\partial \theta} L_1[x(\theta, \chi)], \quad \text{w.p.1, at any } \theta.
\]
Denote the total heights of jumps in \([\theta, \theta + \Delta \theta] \) as
\[
h(\theta, \Delta \theta, \chi) = L_2[x(\theta + \Delta \theta, \chi)] - L_2[x(\theta, \chi)].
\]
Using the fact that \( L_1[x(\theta, \chi)] \) is uniformly differentiable, we have
\[
\frac{\partial J(\theta)}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ E L_1[x(\theta, \chi, \omega, t)] \right]
\]
\[
+ \frac{\partial}{\partial \theta} \left[ E L_2[x(\theta, \chi, \omega, t)] \right]
\]
\[
= E \left( \frac{\partial L_1[x(\theta, \chi, \omega, t)]}{\partial \theta} \right) + \lim_{\Delta \theta \to 0} E \left( \frac{h(\theta, \Delta \theta, \chi)}{\Delta \theta} \right) + \lim_{\Delta \theta \to 0} E \left( \frac{h(\theta, \Delta \theta, \chi)}{\Delta \theta} \right).
\] (16)

Furthermore, \( E[h(\theta, \Delta \theta, \chi)] \) can be factored as
\[
E[h(\theta, \Delta \theta, \chi)] = E[h(\theta, \Delta \theta, \chi)h(\theta, \Delta \theta, \chi) > 0]
\]
\[
\cdot p[h(\theta, \Delta \theta, \chi) > 0]
\]
\[
= H(\theta, \Delta \theta) p[h(\theta, \Delta \theta, \chi) > 0] \] (17)

where \( H(\theta, \Delta \theta) = E[h(\theta, \Delta \theta, \chi)h(\theta, \Delta \theta, \chi) > 0] \) is the mean height of the jumps in \([\theta, \theta + \Delta \theta] \) given that the jump happens and \( p[h(\theta, \Delta \theta, \chi) > 0] \) is the probability that \( L[x(\theta, \chi)] \) jumps in \([\theta, \theta + \Delta \theta] \). Since the probability that \( L[x(\theta, \chi)] \) jumps at \( \theta \) is zero, \( p[h(\theta, \theta, \chi) > 0] = 0 \). Assume that \( \partial p/\partial \Delta \theta \) exists, i.e., the probability that \( L[x(\theta, \chi)] \) jumps in \([\theta, \theta + \Delta \theta] \) changes smoothly as \( \Delta \theta \) changes; then
\[
p[h(\theta, \Delta \theta, \chi) > 0] = g(\theta) \Delta \theta + o(\Delta \theta).
\] (18)

Substituting (17) and (18) into (16) yields
\[
\frac{\partial J(\theta)}{\partial \theta} = E \left( \frac{\partial L_1[x(\theta, \chi, \omega, t)]}{\partial \theta} \right) + H(\theta) g(\theta)
\] (19)
Theorem 1: Estimate 1 converges to \( \partial J(\theta)/\partial \theta \) as \( T \to \infty \) w.p.1 if

1) \( L[x(\theta, \chi(\omega, t))] \) is uniformly differentiable w.p.1 on \( \Omega \); or
2) \( L[x(\theta, \chi(\omega, t))] \) has the form (15) and
   a) \( g(\theta) = 0 \), i.e., the probability that \( L[x(\theta, \chi(\omega, t))] \) jumps in \([\theta, \theta + \Delta \theta] \) is of order \( o(\Delta \theta) \); or
   b) \( H(\theta) = 0 \), i.e., the mean height of jumps at \( \theta \) is zero.

B. Convergence of Estimate 2

Similar to (13), we have

\[
\lim_{T \to (0, \infty)} \left( \frac{\Delta J}{\Delta \theta} \right)_{T, 2} = \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt
\]

\[
= \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt + \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt
\]

\[
= \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt + \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt
\]

\[
= \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt + \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt
\]

\[
= \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt + \lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt
\]

If \( L[x(\theta, \chi(\omega, t))] \) is uniformly differentiable, then (11) holds and

\[
\lim_{T \to (0, \infty)} \left( \frac{\Delta J}{\Delta \theta} \right)_{T, 2} = \frac{\partial J(\theta)}{\partial \theta}
\]

Moreover, for any \( \epsilon > 0 \), we have a \( \delta > 0 \) such that if \(|\Delta \theta| < \delta\), then \(|\partial \theta L[x(\theta, \chi(\omega, t))]/\partial \theta| < \epsilon\) w.p.1 for all \( t \in [0, \infty) \). For any \( \omega \), let \( \Lambda[\chi(\omega, t)] = \{ t \in [0, \infty) : |\partial \theta L[x(\theta, \chi(\omega, t))]/\partial \theta| > \epsilon\} \), and \( m(\Lambda[\chi(\omega, t)]) \) be the Lebesgue measure of \( \Lambda[\chi(\omega, t)] \). Obviously, \( m(\Lambda[\chi(\omega, t)]) \) is a random variable. It is proved in the Appendix that \( m(\Lambda[\chi(\omega, t)]) = 0 \) w.p.1. Therefore, if \(|\Delta \theta| < \delta\), then for all finite \( T \),

\[
\left| \frac{1}{T} \int_0^T \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} dt \right| < \epsilon, \quad \text{w.p.1.}
\]

From this,

\[
\lim_{T \to (0, \infty)} \frac{1}{T} \int_0^T \left| \frac{\partial \theta L[x(\theta, \chi(\omega, t))]}{\partial \theta} \right| dt = 0, \quad \text{w.p.1.}
\]

Therefore, for uniformly differentiable functions, estimate 2 converges to \( \partial J(\theta)/\partial \theta \) w.p.1 as \( T \to \infty \) and \( \Delta \theta \to 0 \).

We now consider functions having the form (15). As indicated by (17) the effect of jumps is decomposed into two parts, the probability that \( L[x(\theta, \chi(\omega, t))] \) jumps in \([\theta, \theta + \Delta \theta] \) and the heights of jumps \( h[\theta, \Delta \theta, \chi] \), provided that the jump occurs. We assume that the jump height is independent of the jump event, i.e., the conditional distribution of jump height \( h[\theta, \Delta \theta, \chi] \) given that \( L[x(\theta, \chi(\omega, t))] \) jumps does not depend on whether the jump occurs at some other time.

First let us study the effect of the jump height. We write the conditional correlation function of \( h[\theta, \Delta \theta, \chi] \) as

\[
R(\tau) = E \left[ h(\theta, \Delta \theta, \chi(\omega, t + \tau)) - H(\theta, \Delta \theta) \right]
\]

\[
\times \left\{ h(\theta, \Delta \theta, \chi(\omega, t + \tau)) - H(\theta, \Delta \theta) \right\}
\]

\[
\times \left\{ h(\theta, \Delta \theta, \chi(\omega, t + \tau)) > 0 \wedge h(\theta, \Delta \theta, \chi(\omega, t + \tau)) > 0 \right\}
\]

where

\[
H(\theta, \Delta \theta) = E \left[ h[\theta, \Delta \theta, \chi(\omega, t)] \right] h(\theta, \Delta \theta, \chi(\omega, t)) > 0 \right\}
\]

We assume that, for all \( \Delta \theta \),

\[
\int_0^\infty |R(\tau)| d\tau < R < \infty.
\]

This assumption is reasonable, since it is equivalent to \( \Phi(0) < \infty \), where \( \Phi \) is the spectral density function of the process \( h[\theta, \Delta \theta, \chi] \). Note that the independence of jump event and jump height implies that \( R(t_1 - t_2) \) does not depend on whether \( L[x(\theta, \chi(\omega, t))] \) jumps at \( t_1 \neq t_2, t_1 \).

Next, we consider the correlation of the jump events. For each realization of \( \chi(\omega, t) \), we define \( \Gamma = \{ t \in [0, T] : h[\theta, \Delta \theta, \chi(\omega, t)] > 0 \} \), and

\[
I_T(t) = \begin{cases} 1, & \text{if } t \in \Gamma \\ 0, & \text{if } t \not\in \Gamma \end{cases}
\]

\[
I_T(t, \tau) = \begin{cases} 1, & \text{if } t \in \Gamma \cap \tau \in \Gamma \\ 0, & \text{if } t \not\in \Gamma \cap \tau \not\in \Gamma \end{cases}
\]

Then \( p(t \in \Gamma) = p(I_T(t) = 1) = p(h[\theta, \Delta \theta, \chi] > 0) \). Let

\[
K(t - \tau) = p(t \in \Gamma \cap \tau \in \Gamma) - p(t \in \Gamma).
\]

Since in equilibrium \( p(t \in \Gamma) = p(\tau \in \Gamma) \), we have \( p(t \in \Gamma \cap \tau \in \Gamma) = p(t \in \Gamma) = p(\tau \in \Gamma) = p(t \in \Gamma) = p(\tau \in \Gamma) \). Thus \( K(t - \tau) = K(t - \tau) \).

\[
K(t - \tau) \text{ is the difference between the conditional probability that } h[\theta, \Delta \theta, \chi] \text{ jumps at } t \text{ given that it jumps at } \tau \text{ and the stationary probability that } h[\theta, \Delta \theta, \chi] \text{ jumps at } t.
\]

We assume that for all \( \Delta \theta \),

\[
\int_0^\infty |K(\tau)| d\tau < K < \infty.
\]

This assumption implies that the jump events at \( t \) and \( \tau \) are asymptotically independent as \( t - \tau \to \infty \).
For functions of the form (15), estimate 2 can be written as

$$
\left( \frac{\Delta J}{\Delta \theta} \right)_{T,2} = \frac{1}{T} \int_0^T \frac{\partial L [x(\theta, \chi(\omega, t))]}{\partial \theta} \, dt \\
+ \frac{1}{T} \int_0^T r_1 \left[ \Delta \theta, \chi(\omega, t) \right] \, dt + \frac{1}{T} \int_0^T h \left[ \theta, \Delta \theta, \chi(\omega, t) \right] \, dt
$$

(22)

where $r_1 [\Delta \theta, \chi(\omega, t)]$ is the corresponding term for $L_1 [x(\theta, \chi(\omega, t))]$ in (12). By the ergodic theorem, the first term on the right in (22) converges to $E( \partial L [x(\theta, \chi(\omega, t))] / \partial \theta)$ w.p.1. We also proved that

$$
\lim_{R \to 0} \frac{1}{T} \int_0^T r_1 [\Delta \theta, \chi(\omega, t)] \, dt = 0, \quad \text{w.p.1.}
$$

Now, let

$$
X_T = \frac{1}{T} \int_0^T h \left[ \theta, \Delta \theta, \chi(\omega, t) \right] \, dt.
$$

Then by (17) and (18),

$$
E(X_T) = \frac{1}{T} \int_0^T E \left( h \left[ \theta, \Delta \theta, \chi(\omega, t) \right] \right) \, dt
$$

$$
= E \left( h \left[ \theta, \Delta \theta, \chi(\omega, t) \right] \right) = H(\theta, \Delta \theta) g(\theta) \Delta \theta.
$$

Let $E_{x_T}$ denote conditional expectation given $I_T$, and let $E_T$ be the expectation operator defined on the $\sigma$-field generated by $I_T$. We have

$$
E(X_T - EX_T)^2
$$

$$
= E_{x_T} \left[ \left( X_T - E_{x_T}X_T \right) + \left( E_{x_T}X_T - EX_T \right) \right]^2
$$

$$
= E(X_T - E_{x_T}X_T)^2 + E_{x_T}E_{x_T}X_T - EX_T)^2. \quad \text{(23)}
$$

We have the following results.

**Lemma 1:**

a) $E_{x_T} (X_T - E_{x_T}X_T)^2 = \frac{1}{T} \int_0^T \int_0^T R(t - \tau) I_T (t, \tau) \, d\tau d\tau$;

b) $E(X_T - E_{x_T}X_T)^2 \leq \frac{2p}{T} \int_0^T \left[ 1 - \frac{\tau}{T} \right] R(\tau) \, d\tau$;

c) $E_{x_T} (X_T - E_{x_T}X_T)^2 \leq \frac{[H(\theta, \Delta \theta)]^2}{T} \int_0^T \left[ 1 - \frac{\tau}{T} \right] K(\tau) \, d\tau$.

The proof of this Lemma, which is purely technical, can be found in the Appendix.

Now assume that the second derivative $\frac{\partial^2 p}{\partial (\Delta \theta)^2}$ exists; then

$$
p \left[ h \left[ \theta, \Delta \theta, \chi(\omega, t) \right] > 0 \right]
$$

$$
= g(\theta) \Delta \theta + q(\theta) (\Delta \theta)^2 + o(\Delta \theta)^2.
$$

**Theorem 2:** 1) For uniformly differentiable functions, estimate 2 converges to $\partial J / \partial \theta$ w.p.1 as $T \to \infty$ and $\Delta \theta \to 0$.

2) For functions having the form (15), if conditions (20) and (21) hold and the probability that the sample performance function jumps in $[\theta, \theta + \Delta \theta]$ has a finite first derivative $g(\theta)$ and a finite second derivative $q(\theta)$, then

a) if $g(\theta) \neq 0$, then $(\Delta J / \Delta \theta)_{T,2}$ converges to $\partial J(\theta) / \partial \theta$ in probability as $T \to \infty$ and $\Delta \theta \to 0$ if and only if $T(\Delta \theta) \to 0$;

b) if $g(\theta) = 0$, i.e., the probability that the sample performance function jumps in $[\theta, \theta + \Delta \theta]$ is of order $o(\Delta \theta)$, then $(\Delta J / \Delta \theta)_{T,2}$ converges to $\partial J(\theta) / \partial \theta$ in probability as $T \to \infty$ and $\Delta \theta \to 0$.

**Proof:** 1) This is a summary of the results proved at the beginning of this subsection.

2a) **Sufficiency:** If $g(\theta) > 0$, then from (23) and Lemmas 1-b) and -c) we have

$$
E \left( \frac{1}{\Delta \theta} (X_T - EX_T)^2 \right)
$$

$$
= \frac{1}{(\Delta \theta)^2} E(X_T - EX_T)^2
$$

$$
\leq \frac{2g(\theta)}{T(\Delta \theta)} \int_0^T \left[ 1 - \frac{\tau}{T} \right] R(\tau) \, d\tau
$$

$$
+ \frac{[H(\theta, \Delta \theta)]^2}{T(\Delta \theta)} \int_0^T \left[ 1 - \frac{\tau}{T} \right] K(\tau) \, d\tau
$$

$$
< \frac{2g(\theta)}{T(\Delta \theta)} \int_0^T \left[ R(\tau) \right] \, d\tau + [H(\theta, \Delta \theta)]^2 \int_0^T \left[ K(\tau) \right] \, d\tau
$$

$$
< \frac{2g(\theta)}{T(\Delta \theta)} \left[ R + K [H(\theta, \Delta \theta)]^2 \right].
$$

From this we have

$$
E \left( \frac{1}{\Delta \theta} X_T - H(\theta) g(\theta) \right)^2
$$

$$
= \frac{1}{\Delta \theta} (X_T - EX_T)^2 + H(\theta, \Delta \theta) g(\theta) - H(\theta) g(\theta)
$$

$$
< \frac{2g(\theta)}{T(\Delta \theta)} \left[ R + K [H(\theta, \Delta \theta)]^2 \right]
$$

$$
+ \left[ g(\theta) [H(\theta, \Delta \theta) - H(\theta)] \right]^2.
$$

The second term on the right side goes to zero as $\Delta \theta \to 0$.

By Chebychev's inequality, $(1/\Delta \theta) X_T$ converges to $H(\theta) g(\theta)$ in probability as $T \to \infty$, $\Delta \theta \to 0$, and $T(\Delta \theta) \to \infty$. Then from (22), $(\Delta J / \Delta \theta)_{T,2}$ converges in probability to $E( \partial L [x(\theta, \chi(\omega, t))] / \partial \theta) + H(\theta) g(\theta)$, which equals $\partial J(\theta) / \partial \theta$ (see (19)).
Necessity: From (23) and Lemma 1-a) we have

\[ E \left( X_T - EX_T \right)^2 \geq E_\tau \left[ E_\tau \left[ X_T - EX_T \right]^2 \right] \]

\[ = E_\tau \left[ \frac{1}{T^2} \int_0^T R(t - \tau) I(t, \tau) d\tau \right] \]

\[ = \frac{1}{T^2} \int_0^T \int_0^T R(t - \tau) E_\tau \left[ I(t, \tau) \right] d\tau d\tau \]

\[ = \frac{1}{T^2} \int_0^T \int_0^T R(t - \tau) p \left\{ h[\theta, \Delta \theta, \chi, \omega, \tau] > 0 \right\} d\tau d\tau \]

For simplicity, we will write \( p[h(t) > 0, h(\tau) > 0] \) for \( p[h[\theta, \Delta \theta, \chi, \omega, t] > 0, h[\theta, \Delta \theta, \chi, \omega, \tau] > 0] \). We have \( p[h(t) > 0, h(\tau) > 0] \)

\[ = p[h(\tau) > 0] p[h(t) > 0] \]

Since \( p[h(t) > 0] > 0 \), \( h(t) > 0 \), \( p[h(\tau) > 0] = g(\theta, \Delta \theta) \).

Next, suppose that \( p[h(\tau) > 0] > 0 \) is continuous at \( \tau = t \). Then for any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( |\tau - t| < \delta \) then \( p[h(t) > 0, h(\tau) > 0] > (1 - \epsilon) g(\theta, \Delta \theta) \).

Using the variable transformation \( \tau = t - \tau \) and \( \tau = t + \tau \), we have

\[ E \left( X_T - EX_T \right)^2 \geq \frac{2}{T^2} \int_0^T \int \left[ R(\tau') \right] \]

\[ \cdot \int_{t - \tau}^{t + \tau} \left[ (1 - \epsilon) g(\theta, \Delta \theta) \right] d\tau' \]

\[ \geq \frac{1}{T^2} \int_0^T \left[ R(\tau') \right] \]

\[ \cdot \int_{t - \tau}^{t + \tau} \left[ (1 - \epsilon) g(\theta, \Delta \theta) \right] d\tau' \]

\[ \geq \frac{2}{T^2} \int_0^T R(\tau') \left( 1 - \epsilon \right) g(\theta, \Delta \theta) d\tau' \]

\[ = \frac{2(1 - \epsilon) g(\theta, \Delta \theta)}{T} \int_0^T \int_0^T R(\tau) d\tau. \]

Therefore, the variance of the third term in the right side of (22) is

\[ \left\{ E \left( X_T - EX_T \right)^2 \right\} \]

\[ \geq \frac{2}{T^2} \left( 1 - \epsilon \right) g(\theta) \int_0^T \left( 1 - \frac{\tau}{T} \right) R(\tau) d\tau. \]

Since the term in the braces is positive, this variance goes to zero only if \( T(\Delta \theta) \to \infty \). Therefore, estimate 2 converges in probability only if \( T(\Delta \theta) \to \infty \).

2b) If \( g(\theta) = 0 \), then \( E(X_T) = o(\Delta \theta) \), and

\[ E \left( \frac{1}{\Delta \theta} (X_T - EX_T) \right)^2 \leq \frac{2q(\theta)}{T} \left\{ R + K \left[ H(\theta, \Delta \theta) \right]^2 \right\}. \]

This implies that estimate 2 converges to \( \partial J(\theta)/\partial \theta \) in probability as \( T \to \infty \) and \( \Delta \theta \to 0 \).

C. Convergence of Estimate 3

Estimate 3 can be written as

\[ \left( \begin{array}{c} \Delta J \\ \Delta \theta \end{array} \right)_{T,3} = \left( \begin{array}{c} \Delta J \\ \Delta \theta \end{array} \right)_{T,2} + \frac{1}{T^2} \int_0^T L \left[ x(\theta, x_2(\omega, \tau)) \right] d\tau \]

\[ - \frac{1}{T^2} \int_0^T L \left[ x(\theta, x_1(\omega, \tau)) \right] d\tau \]

\[ = \left( \begin{array}{c} \Delta J \\ \Delta \theta \end{array} \right)_{T,2} + \frac{1}{T^2} \int_0^T L \left[ x(\theta, x_2(\omega, \tau)) \right] d\tau \]

\[ - EL \left[ x(\theta, x_2(\omega, \tau)) \right] d\tau, \quad (24) \]

where in \( (\Delta J/\Delta \theta)_{T,2} \), \( X_2(\omega, \tau) \) is used as the input random process of \( \left\{ x(\theta, \omega, \tau) \right\} \). Let \( z_1(\theta, \omega, \tau) = L \left[ x(\theta, x_2(\omega, \tau)) \right] \) and \( E \left[ L \left[ x(\theta, x_2(\omega, \tau)) \right] \right] \), \( i = 1, 2 \). Then \( z_1 \) and \( z_2 \) are two independent stochastic processes. Thus

\[ \left[ T \int_0^T z_1(\theta, \omega, \tau) d\tau \right] \]

\[ = \left[ T \int_0^T z_1(\theta, \omega, \tau) d\tau \right] \]

\[ + \left[ T \int_0^T z_2(\theta, \omega, \tau) d\tau \right] \]

Let \( \bar{R}(\tau) = E \left[ z_1(\theta, \omega, \tau) \right] \). Assume that \( \int_0^T \bar{R}(\tau) d\tau < \bar{R} < \infty \). Then

\[ \left[ T \int_0^T \bar{R}(\tau) d\tau \right] \]

\[ = \left[ T \int_0^T \bar{R}(\tau) d\tau \right] \]

Using the same variable transform as for estimate 2, we have

\[ \left[ T \int_0^T \bar{R}(\tau) d\tau \right] \]

\[ = \frac{2}{T^2(\Delta \theta)^2} \int_0^T \bar{R}(\tau) d\tau. \]

A similar result holds for \( z_2(\theta, \omega, \tau) \). These results imply that the second and the third terms on the right side of (24) converge to zero in probability as \( T \to \infty \), \( \Delta \theta \to 0 \), and \( T(\Delta \theta)^2 \to \infty \). Recall that \( (\Delta J/\Delta \theta)_{T,2} \) converges to \( \partial J(\theta)/\partial \theta \) in probability as \( T \to \infty \), \( \Delta \theta \to 0 \), and \( T(\Delta \theta)^2 \to \infty \). We have the following result.

Theorem 3: For functions having the form (15), \( (\Delta J/\Delta \theta)_{T,3} \) converges to \( \partial J(\theta)/\partial \theta \) in probability as \( T \to \infty \), \( \Delta \theta \to 0 \) if and only if \( T(\Delta \theta)^2 \to \infty \).

D. Discussion

Three kinds of estimates were studied in the previous sections. Estimate 1 uses the time average of
with a quadratic performance measure \( L = x^2 \). If \( \alpha \) changes to \( \alpha + \Delta \alpha \), \( x \) will change to \( x' = x + \Delta x \). Thus

\[
dx' = -(\alpha + \Delta \alpha) x'dt + dw. \tag{27}\]

Subtracting (26) from (27) and dividing the result by \( \Delta \alpha \), we have

\[
d\left( \frac{\Delta x}{\Delta \alpha} \right) = -\alpha \frac{\Delta x}{\Delta \alpha} - x'dt. \tag{28}\]

On the other hand, differentiating both sides of (26) with respect to \( \alpha \) yields

\[
d\left( \frac{\partial x}{\partial \alpha} \right) = -\frac{\partial x}{\partial \alpha} dt - xd\alpha. \tag{29}\]

In a vector form, (26)–(29) can be written as

\[
dx = Ax dt + g dw\]

where \( x = (x, x', \Delta x/\Delta \alpha, \partial x/\partial \alpha)^T \), \( g = (1, 1, 0, 0)^T \), and

\[
A = \begin{bmatrix}
-\alpha & 0 & 0 & 0 \\
0 & -(\alpha + \Delta \alpha) & 0 & 0 \\
0 & -1 & -\alpha & 0 \\
-1 & 0 & 0 & -\alpha
\end{bmatrix}
\]

Also,

\[
f(x) = Ax = \begin{bmatrix}
-ax, -(\alpha + \Delta \alpha)x', -\alpha \frac{\Delta x}{\Delta \alpha} - x', -ax - x
\end{bmatrix}^T.
\]

Since \( L = x^2 \), we have

\[
\frac{\Delta x}{\Delta \alpha} = \frac{(x + \Delta x)^2 - x^2}{\Delta \alpha} = \frac{2x(\Delta x/\Delta \alpha - \frac{\partial x}{\partial \alpha})}{\Delta \alpha} \tag{30}\]

(higher order terms are omitted in (30)).

Let \( \Psi_1 = x(\Delta x/\Delta \alpha) \). It is easy to check that \( g^T(\partial^2 \Psi_1/\partial \tilde{x}^2)g = 0 \). Then by applying (25), we get

\[
d\left\{ E \left[ x^T \Delta x \right] / \Delta \alpha \right\} E \left[ f^T(x) \partial \Psi_1 / \partial \tilde{x} \right] dt
\]

\[
= -E \left[ \alpha \frac{\Delta x}{\alpha} + \frac{\Delta x}{\Delta \alpha} + ax \right] dt.
\]

For a stationary process, \( d\{ E(x(\Delta x/\Delta \alpha)) \} = 0 \). Therefore,

\[
2\alpha \times E \left[ x^T \Delta x \right] / \Delta \alpha = -E(\chi x').
\]

Now let \( \Psi_2 = x(\partial x/\partial \alpha) \). By the same derivation, we have

\[
2\alpha \times E \left[ x \partial x / \partial \alpha \right] = -E(x^2).
\]

Thus

\[
E \left[ \frac{\Delta x \chi(\omega, t)}{\Delta \alpha} \right] = \frac{1}{\alpha} (E(x^2) - E(\chi x')). \tag{31}\]

IV. An Example

A stochastic system is often described by the Ito stochastic differential equation

\[
dx = f(x, \theta, t) dt + g(x, \theta, t) dw,
\]

in which \( w \) is a standard Wiener process, \( x \in R^n \) and \( f \) and \( g \) are \( n \)-dimensional functions. The probability density function \( p(t, x) \) of the system satisfies the Fokker-Planck equation

\[
\frac{\partial p(t, x)}{\partial t} = -\sum_{i=1}^n \frac{\partial}{\partial x_i} \left[ f_i(x, \theta, t) p \right] + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \left[ g_i(x, \theta, t) g_j(x, \theta, t) p \right].
\]

Letting the right side of this equation equal zero, we get an equation for the invariant probability density function. A system starting with this invariant probability as the probability of the initial state is stationary. The expectation of any differentiable function \( \Psi(x) \) satisfies

\[
d\{ E[\Psi(x)] \}
\]

\[
e \left[ f^T(x, \theta, t) \frac{\partial \Psi}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2} g(x, \theta, t) \right] dt
\]

(25)

where the superscript \( T \) denotes transpose, \( \partial^2 \Psi / \partial x^2 \) is the Jacobi matrix of \( \Psi \), and \( \partial \Psi / \partial x \) is an \( n \)-dimensional vector.

As an example, we consider a one-dimensional linear system

\[
dx = -ax dt + dw
\]

(26)
Using (25) and (26) for $\Psi(x) = x^2$, we have
$$d\{E(x^2)\} = E\{-2\alpha x^2 + 1\} \, dt.$$ 
Again we have $dE(x^2) = 0$ for stationary processes. Thus
$$E(x^2) = \frac{1}{2\alpha}.$$  
(32)

Similarly, from (26) and (27) we get
$$d\{E(xx')\} = E\{-\alpha xx' - (\alpha + \Delta \alpha) xx' + 1\} \, dt.$$ 
In steady state
$$E(xx') = \frac{1}{2\alpha + \Delta \alpha}.$$  
(33)

From (31)–(33), we finally obtain
$$\lim_{\Delta \alpha \to 0} E\left\{\frac{\Delta \alpha, x(x, \omega, t)}{\Delta \alpha}\right\} = \lim_{\Delta \alpha \to 0} \frac{1}{\Delta \alpha} \left\{\frac{1}{2\alpha} - \frac{1}{2\alpha + \Delta \alpha}\right\} = 0.$$ 

This implies that $(\partial J/\partial \alpha) J(x(\theta, \chi(\omega, t)))$ converges to $(\partial J/\partial \alpha)$ with probability one as $T \to \infty$.

The estimator of $\partial J/\partial \alpha$ based on (26) and (29) is shown in Fig. 2. Although for a linear system the structure of this estimator is about the same as that of the system, it still has advantages over the crude Monte Carlo estimates. First, for practical systems, the white noise $d\omega$ is usually not repeatable. Thus estimate 3 has to be used. Estimate 3 has a large error since it estimates the difference of the mean of two random variables divided by a small number. Secondly, estimate 1 gives directly the value of $\partial J(\theta)/\partial \theta$ as $T \to \infty$, while estimate 2 yields $(\Delta J/\Delta \alpha)$ as $T \to \infty$ for a fixed $\Delta \alpha$, which is a biased estimate of $\partial J(\theta)/\partial \theta$.

In principle, a similar derivation (albeit more complicated) applies to multidimensional linear systems with quadratic performance functions. To decide whether estimate 1 is unbiased for a nonlinear system requires further research.

V. CONCLUSION

An estimate (estimate 1) of the performance sensitivity $\partial J(\theta)/\partial \theta$ using the time average of the sample derivative $(\partial J(\theta)/\partial \theta) J(x(\theta, \chi(\omega, t)))$ is proposed. For uniformly differentiable functions or functions which jump in $[\theta, \theta + \Delta \theta]$ with probability $o(\Delta \theta)$, the interchangeability of the two operators $\partial/\partial \theta$ and $E$ holds, and this estimate converges to $\partial J(\theta)/\partial \theta$ with probability one as $T \to \infty$. This estimate requires only one simulation.

The convergence properties of two crude Monte Carlo estimates, estimates 2 and 3, are also studied. Estimate 2 uses the same input random process while estimate 3 uses two different input processes. Generally, the variance of estimate 2 is of order $O(1/T(\Delta \theta)^2)$ and that of estimate 3 is of order $O(1/T(\Delta \theta)^3)$. Each of these two estimates requires two simulations.

If the interchangeability (11) holds, estimate 1 is the best among these three estimates. This estimate is unbiased even for finite $T$, and its variance depends only on $T$. For nonlinear systems, the structure of the estimate may be much simpler than that of the system. Therefore, it may save considerably on computations in simulation.

If the interchangeability (11) does not hold, then estimate 2 or 3 must be used. Estimate 2 has less variance but requires the same input process which may not be feasible in practice.

As an example, we proved that for a one-dimensional linear system with a quadratic performance function the interchangeability (11) holds. The technique used in the example applies to multidimensional linear systems with quadratic performance functions.

APPENDIX

A. Proof of $m(\Lambda(\chi(\omega, t))) = 0$ w.p.1

Since $L[x(\theta + \Delta \theta, \chi(\omega, t))], L[x(\theta, \chi(\omega, t))], \text{ and (}\partial J(\theta)/\partial \theta) J(x(\theta, \chi(\omega, t)))$ are measurable stochastic processes, $r(\Delta \theta, \chi(\omega, t))/\Delta \theta$ is measurable. Let $\Pi = \{I(\omega, t) \in \Omega \times R^1 : |r(\Delta \theta, \chi(\omega, t))/\Delta \theta| > \epsilon\}$. Then $\Pi \subset \Omega \times R^1$ is a $F \times B$ measurable set, where $B$ is the $\sigma$-field of Borel subsets of $R^1$. Let $L_{\Pi}(\omega, t)$ be the indicator function of $\Pi$, i.e., $L_{\Pi}(\omega, t) = 1$, if $(\omega, t) \in \Pi$; 0, if not. Then $\int_{\Omega \times R^1} L_{\Pi}(\omega, t) \, dI = m(\Lambda(\chi(\omega, t)))$. Furthermore, by Fubini’s theorem [14], the measure of $\Pi$ is

$$m(\Pi) = \int \int_{\Omega \times R^1} L_{\Pi}(\omega, t) \, dI(\omega, t) = \int \int_{\Omega \times R^1} m(\Lambda(\chi(\omega, t))) \, dI(\omega, t).$$

Since $\int_{\Omega \times R^1} L_{\Pi}(\omega, t) \, dI = 0$, we have

$$\int \int_{\Omega \times R^1} L_{\Pi}(\omega, t) \, dI(\omega, t) = \int \int_{\Omega \times R^1} m(\Lambda(\chi(\omega, t))) \, dI(\omega, t) = 0.$$  

Therefore, $p\{m(\Lambda(\chi(\omega, t))) = 0\} = 0$, or $m(\Lambda(\chi(\omega, t))) = 0$ w.p.1.

B. Proof of Lemma 1

Part a)

$$E_{\psi} E_{\psi}(X_T - E_{\psi} X_T)^2$$

$$= E_{\psi} E_{\psi}(X_T^2) - E_{\psi} E_{\psi}(X_T)^2$$

$$= \frac{1}{T^2} E_{\psi} \left\{ \int_0^T \left[ \int_0^T h(\theta, \Delta \theta, \chi(\omega, t)) \right] dt - \left[ \int_0^T h(\theta, \Delta \theta, \chi(\omega, t)) dt \right]^2 \right\}.$$
\[ E_{\tau} \left[ I_{\tau} \left\{ h(\theta, \Delta \theta, x(\omega, t)) > 0 \right\} \right] \]

\[ = \frac{1}{T^2} \mathbb{E} \left\{ \int_{T/2}^{T} \left[ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right] \, dt \right\} \]

\[ \leq \left\{ \frac{1}{T^2} \mathbb{E} \left\{ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right\} \right\}^2 \]

\[ = \left\{ \frac{1}{T^2} \int_{T/2}^{T} E_{\tau} \left[ h(\theta, \Delta \theta, x(\omega, t)) \right] \, dt \right\} \]

\[ \leq \left\{ \frac{1}{T^2} \int_{T/2}^{T} \left[ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right] \right\} \]

\[ = \frac{1}{T^2} \left\{ \int_{T/2}^{T} \left[ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right] \right\} \]

\[ \leq \frac{2}{T^2} \int_{T/2}^{T} \frac{T}{T} \int_{0}^{T-\tau} \frac{1}{T} \int_{0}^{T} R(\tau) \, d\tau \, d\tau \]

\[ = \frac{2}{T^2} \int_{T/2}^{T} \frac{T}{T} \int_{0}^{T-\tau} \frac{1}{T} \int_{0}^{T} R(\tau) \, d\tau \, d\tau \]

\[ \text{Part c)} \]

\[ E_{\tau} \left[ X_{\tau} \right] = E_{\tau} \left\{ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right\} \]

\[ = \left\{ \frac{1}{T^2} \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \right\} \]

\[ = \frac{1}{T^2} \int_{T/2}^{T} E_{\tau} h(\theta, \Delta \theta, x(\omega, t)) \]

\[ \text{where } m(\Gamma) \text{ is the Lebesgue measure of set } \Gamma. \]

\[ E_{\tau} \left[ E_{\tau} X_{\tau} - EX_{\tau} \right] \]

\[ \text{Part b)} \]

\[ E_{\tau} \left[ (X_{\tau} - E_{\tau} X_{\tau})^2 \right] \]

\[ = E_{\tau} \left\{ \int_{T/2}^{T} I_{\tau} \left( h(\theta, \Delta \theta, x(\omega, t)) > 0 \right) \, dt \right\} \]

\[ = \frac{1}{T^2} \int_{T/2}^{T} \int_{T/2}^{T} \left\{ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right\} \]

\[ = \frac{1}{T^2} \int_{T/2}^{T} \int_{T/2}^{T} \left\{ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right\} \]

\[ = \frac{1}{T^2} \int_{T/2}^{T} \int_{T/2}^{T} \left\{ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right\} \]

\[ = \frac{1}{T^2} \int_{T/2}^{T} \int_{T/2}^{T} \left\{ \int_{T/2}^{T} h(\theta, \Delta \theta, x(\omega, t)) \, dt \right\} \]

\[ \text{using the transformation } \tau = t - \tau, \] and \( \tau = t + \tau, \] we can prove

\[ \frac{1}{T^2} \int_{T/2}^{T} \int_{T/2}^{T} K(t - \tau) \, dt \, d\tau \]

\[ = \frac{1}{T^2} \int_{T/2}^{T} \int_{T/2}^{T} K(t - \tau) \, dt \, d\tau \]

\[ = \frac{1}{T^2} \int_{T/2}^{T} \int_{T/2}^{T} K(t - \tau) \, dt \, d\tau \]

\[ = \frac{1}{T^2} \int_{T/2}^{T} \int_{T/2}^{T} K(t - \tau) \, dt \, d\tau \]
REFERENCES