State Aggregation and Discrete-State Markov Chains Embedded in Point Processes

Xi-Ren Cao
Digital Equipment Corporation
305 Foster Street, Littleton, MA 01460

Abstract

One result that is of both theoretic and practical importance regarding point processes is the method of thinning. The basic idea of this method is that under some conditions, there exists an embedded Poisson process in any point process such that all its arrival points form a subsequence of the Poisson process. We extend this result by showing that on the embedded Poisson process one can define a Markov chain with a discrete state that characterizes the stage of the interarrival times. This implies that one can construct embedded Markov chains with countable state spaces for the state processes of many practical systems that can be modeled by point processes.

1. Introduction

Point-process models are widely used in many theoretical and practical fields, ranging from statistical physics, queueing network theory, to performance analysis of computer-communication networks (Brémaud 1981). One of the important results regarding point processes is the “thinning” (or uniformization) method (see, e.g., Lewis and Shedler 1979, Ogata 1981, Shanthikumar 1988, and Van Dijk 1990). By thinning, one can construct a point process based on a Poisson process.

The thinning method can be briefly explained as follows. Let us first define a (nonexplosive) point process (Brémaud 1981). A realization of a point process over $[0, \infty)$ is a sequence $(T_0, T_1, \ldots)$, with $T_k < T_{k+1}$ and $\lim_{k \to \infty} T_k = +\infty$. The counting process $N_t = k$ if $t \in (T_k, T_{k+1})$, $k \geq 0$, is also called a point process. Thus, a point process can be considered as defined on $[0, \infty)$. A "continuous-time" view of a point process (instead of a sequence of random times) is more suitable for most applications where "physical states" (such as the number of customers in queuing networks) may be defined associated with a point process. Each point $T_k$ can be considered as an arrival instant of an event. Thus, $T_{k+1} - T_k$ is called an interarrival time of the event. Thinning, or the uniformization method, claims that under some mild conditions, there is a Poisson process $T := \{T_0, T_1, \ldots\}$ such that $T_k \in T$. For convenience, we say that the point process (in the sense of a continuous-time process) has an embedded Poisson process.

The main result of this paper is: on the embedded Poisson process of a point process, we can define a discrete state $u_k$, which characterizes the elapsed time of the interarrival time at $T_k$ so that $\{w_0, w_1, \ldots\}$ is a Markov chain. This allows us to construct an embedded Markov chain with a countable state space for many practical systems whose state processes may be modeled as point processes. Applying the existing results of Markov chains with countable states may yield many useful results for practical systems, including some convergence properties that are otherwise difficult to establish.

The proof of the result is based on a very simple idea, i.e., the state aggregation. While the elapsed time of an event at $T_k$, $u_k$, usually cannot be aggregated into discrete states that form a Markov chain, it turns out that such an aggregation exists in a larger state field that represents an augmentation of the state. In section 2, we discuss the fundamentals for state aggregation of Markov chains. Section 3 discusses the univariable point processes and section 4 the multi-variable ones. An M/G/1 is discussed as a simple example to show the application of this result.

2. Aggregation on an augmented state space

Consider a time-homogeneous Markov chain $S = \{s_0, s_1, \ldots\}$ with a state space $S$, which may consist both discrete and continuous components. We use the notation $P(s'|s)$ to quantify the state transition of the Markov chain. For discrete states, it is the conditional probability, and for continuous states, it is the conditional probability density. For example, suppose that $s = (x, t)$ is a state with $x$ being the discrete part and $t$ the continuous part. Then $P(s'|s) = P(s'|s)$ is the conditional probability of $\{x_{k+1} = x'\}$ and $\{T_{k+1} \in (t', t' + dt')\}$, given $s_t = (x, t)$. A similar notation is used for conditional probabilities. We use $\sum_{s'} P(s'|s) = 1$ to denote the normalizing condition. Note that the summation has to be replaced by an integration for continuous states; this applies to all the other summations over states in this paper.

Now consider a partition of the state space $S$: $\{S_1, S_2, \ldots\}$, with $\bigcup S_i = S$ and $S_i \cap S_j = \emptyset$, $i \neq j$. Each set in the partition can be considered as a distinct state. With Markov chain $S$, we can define a stochastic process $W = \{w_0, w_1, \ldots\}$ as follows: if $s_k \in S_j$ then set $w_k = j$. Thus, $W$ is a process defined on the countable
space \( \{1, 2, \ldots\} \), which may be finite or infinite. However, \( W \) may not be a Markov chain.

The condition for \( W \) to be a Markov chain can be easily found (see, e.g., Kemeny and Snell 1960). We have

\[
P(w_{k+1}|w_k = i) = \sum_{s \in S} P(w_{k+1}|s = s)P(s = s|s_k \in S_i),
\]

and

\[
P(w_{k+1}|w_k = i, w_{k-1} = j) = \sum_{s \in S} P(w_{k+1}|s = s)P(s = s|s_k \in S_i, s_{k-1} \in S_j).
\]

Thus, a sufficient condition for \( W \) to be Markov, i.e., \( P(w_{k+1}|w_k = i, w_{k-1} = j) = P(w_{k+1}|w_k = i) \), is

\[
P(s_k = s|s_k \in S_i) = P(s_k = s|s_k \in S_i, s_{k-1} \in S_j)
\]

for all \( i, j \). In words, this says that knowing the set in which \( s_k \) is, does not help to determine the particular state in \( S_i \). It is easy to see that (1) is equivalent to the following condition:

\[
P(s_{k-1} \in S_j|s_k = s) = P(s_{k-1} \in S_j|s_k \in S_i)
\]

all \( i, j \), (2)

or,

\[
P(s_{k-1} \in S_j|s_k = s) = \text{const.}
\]

for all \( s \in S_i \), all \( i, j \). (3)

In words, knowing the set in which \( s_k \) is, one cannot obtain better knowledge about the set in which \( s_{k-1} \) is by looking at which state \( s_k \) is.

Condition (1) or (2) or (3) is very restrictive and many practical systems do not satisfy this condition. Perhaps for most systems with continuous states the only partition that leads to Markov aggregation is a singleton, i.e., \( S_i = S \) and \( w_k \equiv 1 \) for all \( k \). We call this a trivial aggregation. This trivial aggregation does not yield any information.

The main idea of this paper is that we can augment the state space so that for the augmented state space there exist some non-trivial partitions for which the aggregated processes are Markov chains. Let us state this idea in a more formal setting.

Let \((\Omega, \Sigma, P)\) be the probability space on which the Markov chain \( S \) is defined. Let \( \Theta \) be the \( \sigma \) field generated by \( S \), i.e., the smallest \( \sigma \) field on which \( S \) is measurable. Then \( s_k, k = 0, 1, \ldots, \) are \( \Theta \)-measurable random variables. Suppose that there is another sequence of random variables, \( \xi_0, \xi_1, \ldots, \) that are measurable with respect to another \( \sigma \) field \( \Xi \). Let \( \mathcal{R} \) be the state space of this sequence. Then \( \xi_0, \xi_1, \ldots \) is defined on \((\Omega, \Xi, P)\). Let \( u_k = (s_k, \xi_k) \) be the augmented state. We assume that \( U = \{u_0, u_1, \ldots\} \) is a Markov chain. \( U \) is defined on \((\Omega, \Sigma, P)\), where \( \Sigma = \sigma(\Theta, \Xi) \) is the \( \sigma \) field generated by \( \Theta \) and \( \Xi \). The state space of \( U \) is \( \mathcal{D} = S \times \mathcal{R} \).

The \( \sigma \) field \( \Xi \) is usually finer than \( \Theta \), i.e., the Markov chain \( U \) usually contains more information than \( S \). An important observation is that even if \( S \) only has a trivial Markov aggregation, \( U \) may have non-trivial Markov aggregation for properly chosen \( \Xi \). We shall see that this aggregation on the augmented space may lead to some advantages.

Now, consider a partition of \( \mathcal{D} = \cup D_i \). The augmented Markov chain \( U \) may be aggregated into a chain \( V \). From (1), the condition for \( V \) to be Markov is

\[
P(s_k = s, \xi_k = \xi | u_k \in D_i)
\]

\[
= P(s_k = s, \xi_k = \xi | u_k \in D_i, u_{k-1} \in D_j)
\]

for all \( i, j, k \). If \( \xi_k \) and \( s_k \) are independent, then condition (4) becomes

\[
P(s_k = s | u_k \in D_i)
\]

\[
= P(s_k = s | u_k \in D_i, u_{k-1} \in D_j)
\]

and

\[
P(\xi_k = \xi | u_k \in D_i)
\]

\[
= P(\xi_k = \xi | u_k \in D_i, u_{k-1} \in D_j)
\]

(5)

Conditions (5) and (6) will be used in the rest of this paper to prove that the aggregated chains are indeed Markov.

3. Discrete-state embedded Markov chains in univariate point processes

In this section, we shall show that under some conditions, there exits an embedded Markov chain with a discrete state for any univariable point process. The discrete state represents the "stage" of the interarrival time. We first illustrate this idea by using a simple point process, i.e., the Poisson process.

Example 1. Consider a Poisson process \( \{T_0, T_1, \ldots\} \); its interarrival time is a random variable having an exponential distribution with mean \( 1/\lambda_0 \). For any \( 0 < p < 1 \), let \( \lambda = \frac{\lambda_0}{p} \) and we can write the probability density function as

\[
f(r) = \lambda_0 e^{-\lambda_0 r} = \sum_{n=1}^{\infty} p(1-p)^{n-1} \frac{\lambda^r}{(n-1)!}.
\]

This shows that the random variable can be represented by the service time of an exponential server with a mean service time \( 1/\lambda \) and a feedback rate \( (1-p) \). At the service completion time at the server, a customer leaves the server with probability \( p \) and goes back to the server for another service with probability \( (1-p) \). We call each
of these service periods a stage. Now consider the inter-arrival time of the Poisson process as the lifetime of an event. An arrival instant, a new lifetime starts and the previous one ends. Based on (7), each lifetime can be considered as the sum of $n$ exponential random variables with an equal mean $1/\lambda$, with $n$ being a random number. Thus, the instants at which the lifetimes transit to the next stage form a Poisson process with rate $\lambda$, denoted as $\{t_0, t_1, \ldots\}$.

Let $s(t)$ be the elapsed time of the current event at time $t$ (the time elapsed since the latest arrival instant). Assume that $s(t)$ is right continuous. Let $s_k := s(t_k)$. Then $\{s_0, s_1, \ldots\}$ is a Markov chain with state space $[0, \infty)$. Given $s_k$, $s_{k+1} = 0$ (a new event starts) with probability $p$, and $s_{k+1} \in [s_k + \tau, s_k + \tau + dr]$ with probability $(1 - p)\lambda e^{-\lambda \tau} dr$. It is clear that this Markov chain has no non-trivial aggregations. Now let us augment the state space and use a finer $\sigma$ field. First, at each $t_k$ there is a uniformly distributed random variable $\eta_k \in [0, 1]$ that determines if the lifetime terminates at $t_k$ in the following way: if $\eta_k < p$ then it terminates, otherwise it goes to the next stage. Let $\xi_k = (\eta_k, \eta_{k-1}, \ldots)$. We attach $\xi_k$ to $s_k$ to make a bigger state $u_k = (s_k, \xi_k)$. $U = \{u_0, u_1, \ldots\}$ is a Markov process. The state space of $U$ is $D = [0, \infty) \times [0, 1)^\infty$.

Now we partition the augmented state space $D$. Let

$$D_1 = \{(s_k, \xi_k) \in D : s_k \in [0, \infty), \eta_k \in (0, p)\},$$

$$D_2 = \{(s_k, \xi_k) \in D : s_k \in [0, \infty), \eta_k \in [p, 1 - p), \eta_{k-1} \in (0, p)\},$$

$$\ldots$$

$$D_n = \{(s_k, \xi_k) \in D : s_k \in [0, \infty), \eta_{k-i} \in (1 - p, 1), i = 0, \ldots, n-2, \text{ and } \eta_{k-n+1} \in (0, p)\},$$

$$\ldots$$

If $(s_k, \xi_k) \in D_n$, then $s_k$ is in its $n$th stage. $\xi_k$ can be chosen as a vector with a finite number of components: $\xi_k = (\eta_k, \ldots, \eta_{k-n+1})$. In the aggregated chain $V = \{v_0, v_1, \ldots\}$, $v_k$ is in fact the stage of $s_k$. Since knowing the stage of the previous state does not help to know the value of $s_k$ and $\xi_k$, (5) and (6) hold for $V$. Thus, $V$ is a Markov chain.

This example shows that although we cannot partition the range of $s$ to aggregate the Markov chain $\{s_0, s_1, \ldots\}$, we can decompose the event lifetime into stages and obtain an aggregated Markov chain. Using the decomposition, we can see that each lifetime belongs to an $n$-stage Erlang distribution with mean $\frac{1}{\lambda} = \frac{p}{\lambda}$ for some $n$. The decomposition is finer when $p$ is smaller.

The example shows that the decomposition of an exponential random variable enables us to obtaining an aggregated Markov chain. Next, we show that a decomposition procedure similar to the exponential case can be developed for random variables with a large class of general distributions. The procedure is based on a random variable generation method, which was introduced by Shantihkumar (1989), and was further studied in Cao (1992). Using the decomposition result, we can specify the elapsed time of an event by its stage.

Let $R$ be a random variable, which can be considered as an event lifetime, and $F(r)$ be its cumulative probability distribution function. Assume that the probability density function $f(r) = \frac{\lambda}{r} F'(r)$ and the hazard rate function

$$\alpha(r) = \frac{f(r)}{1 - F(r)} \quad r > 0$$

exist and $\sup(\alpha(r), r > 0) < \lambda < \infty$. (Thus, $F(r) \neq 1$ for all $0 < r < \infty$.)

The idea of random variable decomposition can be explained by using Figure 1. In the figure, each circle is a stage representing an exponentially distributed random variable with mean $1/\lambda$. An event starts with entering stage 1. At the end of stage $n$, the event either (a) terminates, which has probability $p_n := \alpha^n/\lambda$ and enters the next stage, or (b) it continues, which has probability $1 - p_n$, where $r_n$ is the elapsed time of the event at the end of stage $n$. It has been shown that the lifetime of the event thus generated has a distribution function $F(r)$ (see Proposition 1 below). In fact, this random variable decomposition is the foundation of the thinning (or uniformisation) method for point processes (Lewis and Shedler 1979, Ogata 1981).

Let $P(R = r, K = n)dr$ be the probability that the event terminates at the end of the $n$th stage and that $r_n = R \in [r, r + dr)$. The following proposition is proved in Cao (1992).

**Proposition 1.** Let $\lambda > \sup(\alpha(r), r > 0)$. Then

$$f(r) = \alpha(r)e^{-\int_0^r \alpha(\tau) d\tau} = \sum_{n=1}^{\infty} \alpha_n(r),$$

\[ (8) \]
where
\[ \varphi_n(r) = P(R = r, K = n) \quad n \geq 1. \] (9)

The explicit form of \( \varphi_n(r) \) can be easily derived by virtue of Figure 1. For example,
\[ \varphi_2(r) = \int_0^\infty \left( \lambda e^{-\lambda r} \left[ 1 - \frac{\alpha(r_1)}{\lambda} \right] \int_{r_1}^\infty \left( 1 - \frac{\alpha(r_2)}{\lambda} \right) \lambda e^{-\lambda (r_2 - r_1)} \right) \lambda e^{-\lambda r_2} \frac{\alpha(r_2)}{\lambda} dr_1 dr_2. \] (10)

\( \varphi_k(r) \) can be obtained by a set of recursive equations presented in (Cox 1992).

Equations (8)-(10) are similar to (7), except that the terminating probability is \( p = \alpha(\cdot)/\lambda \), which depends on the elapsed time \( r \). If the number of stages is finite and \( \alpha(r_n)/\lambda \) equals a constant \( p_n \) for each \( n \), the decomposition reduces to the well-known Coxian distribution (Kleinrock 1975). If furthermore \( p_n = p \) for all \( n \), then we get the exponential case shown in (7).

Now, consider a univariate Poisson process whose event interarrival time has a general distribution \( F(r) \). From Proposition 1, similar to the case studied in Example 1, there is an embedded Poisson process \( \{t_0, t_1, \ldots \} \) with interarrival rate \( \lambda \). Again, we define \( s_k = s(t_k) \) as the elapsed time of the event at \( t_k \). Then \( \{s_0, s_1, \ldots \} \) is a Markov chain with state space \([0, \infty)\); given \( s_n, s_{n+1} = 0 \) with probability \( p_n := \int_{0}^{\infty} \alpha(s_n + r) e^{-\lambda r} dr \), and \( s_{n+1} \in \{s_n + r, s_n + r + \ldots \} \) with probability \( 1 - \alpha(s_n + r) e^{-\lambda r} dr \). Because \( p_n \) depends on \( s_n \), the augmented state takes the form \( u_n = (s_n, s_{n-1}, \ldots, s_{n-k+1}; \eta_n, \eta_{n-1}, \ldots, \eta_{n-k+1}) \). The augmented state space is \( D = [0, \infty)^m \times [0, 1]^m \) with a partition \( D = \cup_{n=1}^{\infty} D_n \), where
\[ D_n = \{ u_k \in D : u_{k-j} \in [0, \infty), j = 0, \ldots, n-1, \]
\[ \text{and } u_{k-n-j} \in [0, p_n - p_{n-j}) \}, \]
\[ n \in \mathbb{N} \]. (11)

If \( u_n \in D_n \), then \( s_n \) is in its \( n \)th stage. Again, knowing the stage of the previous state does not help to know the value of \( (s_{\pi}, s_{\pi-1}, \ldots, s_{\pi-n+1}) \) and \( (\eta_{\pi}, \eta_{\pi-1}, \ldots, \eta_{\pi-n+1}) \). That is, (5) and (6) hold for the aggregated chain \( V \), with \( (s_\pi, \ldots, s_{\pi-n+1}) \) and \( (\eta_\pi, \ldots, \eta_{\pi-n+1}) \) considered as two independent random vectors. (If \( u_n \in D_n \) and \( n > 1 \), then \( u_{n-1} \in D_{n-1} \) with probability one (w.p.1); if \( u_n \in D_1 \), then \( s_0 = 0 \) w.p.1.) Thus, \( V \) is a Markov chain.

Finally, let \( G(t) := u_k \) for \( t \in [t_k, t_{k+1}) \), we obtain a pure jump process whose state is the event. \( G(t) \) jumps at the arrival instants of a Poisson process and has an embedded Markov chain \( \{v_0, v_1, \ldots \} \). However, one should note that \( G(t) \) is not a semi-Markov process, since the transition probability of \( G(t) \) at \( t_k \) depends on not only \( t_k - t_{k-1} \), but also the previous interarrival times.

4. Multi-variable point processes

In Section 3, we discussed Poisson processes and other univariable point processes. However, most systems have more than one event, corresponding to multivariable point processes. In this section, we study two-variable point processes, the principle for more variables is just the same.

The points associated with each variable can be considered as a univariable point process whose interarrival times are the lifetimes of an event. Let \( \alpha_1 \) and \( \alpha_2 \) be the hazard rates of the two event lifetimes, and \( \lambda_1 \) and \( \lambda_2 \) be the upper bounds of these two hazard rates, respectively. Let \( T_1 := \{s_1^{(1)}, s_1^{(2)}\} \) and \( T_2 := \{s_2^{(1)}, s_2^{(2)}\} \) be the embedded Poisson processes in these two point processes. The composition of these two Poisson processes, denoted as \( T := \{t_0, t_1, \ldots \} \), is a Poisson process with arrival rate \( \lambda = \lambda_1 + \lambda_2 \). A point in \( T \) is in \( T_1 \), with probability \( \lambda_1/\lambda \), \( i = 1, 2 \).

Let \( s_i := (s_i^{(1)}, s_i^{(2)}) \), where \( s_i^{(i)} \), \( i = 1, 2 \), is the elapsed time of event \( i \) at time \( t_i \), and let \( y_i := (y_i^{(1)}, y_i^{(2)}) \), where \( y_i^{(i)} \), \( i = 1, 2 \), is the stage of \( s_i^{(i)} \). There is one major difference between the single event and the multi-event cases: In the single event case, knowing the stage of \( s_1 \) is equivalent to knowing all the previous stages up to the start of the event lifetime, e.g., if \( y_1 = n \), then \( y_2 = n \). In the two-event case, given \( y_1 = n \), one may have either \( y_2 = n - 1 \) or \( y_2 = n \), depending on whether \( t_2 \) is in \( T_1 \) or \( T_2 \). Thus, \( y_2 \) does not completely represent the history. In other words, knowing \( y_{k-1} \) helps to determine \( s_k \). For example, consider \( y_2 = (1, 4), y_1 = (6, 4) \) and \( y_3 = (1, 3) \). \( y_2 \) implies that \( s_2 \) is associated with the first event and is the first time that the event enters stage 1 (i.e., \( s_2^{(1)} = 0 \)), while \( y_3 \) implies that \( s_3 \) is associated with the second event and that at \( t_3 \), the first event has been in stage 1 for at least one interarrival period of \( T \). Thus, the previous stage \( y_{k-1} \) does give information about how long the event has been in its current stage.

The above analysis indicates that a partition similar to (11) would not work in the multi-event case. In fact, (11) does not describe how process \( T \) is decomposed into \( T_1 \) and \( T_2 \). Thus, a larger augmented space is needed. Let \( u_n := (s_n, s_{n-1}, \ldots, s_{n-k+1}; \eta_n, \eta_{n-1}, \ldots, \eta_{n-k+1}) \), with \( s_j := (s_j^{(1)}, s_j^{(2)}), j = k, k - 1, \ldots, k - n + 1 \), and \( \eta_j := (\eta_j^{(1)}, \eta_j^{(2)}), j = k, k - 1, \ldots, k - n + 1 \), being random variables uniformly distributed on \([0, 1] \). If \( \xi_i \in [0, 1] \), then \( t_i \) is in \( T_1 \) and let \( i = 1 \), otherwise \( t_i \) is in \( T_2 \) and \( i = 2 \). The augmented state space is \( D = [0, \infty)^m \times [0, 1)^m \times (0, 1)^m \) with a partition \( D = \cup_{j=1}^{\infty} D_j \), where \( z \) is a discrete index and \( D \) is determined as follows. Let \( K_1 := \{k_1^{(1)}, k_1^{(2)} \} \) and \( K_2 := \{k_2^{(1)}, k_2^{(2)} \} \) be a partition of the natural numbers, \( K_1 \cup K_2 = \{0, 1, \ldots \} \) and \( K_1 \cap K_2 = \emptyset \). If \( j \in K_1 \), then \( t_j \in T_1 \) and set \( i_j = 1 \); if \( j \in K_2 \), then \( t_j \in T_2 \) and set \( i_j = 2 \);
Let $p_{r-j} = \frac{\alpha(\frac{\nu_j}{\nu_j + \rho})}{\lambda}$. Define

$$D_i = \{u_i \in D : s_{a-i} \in [0, \infty), i = 1, 2, \}$$

for $h - j \in K_1 : z_{a-j} \in [0, 1), \eta_{b-r} \in [0, \eta_{b+n-1}], j = 0, 1, \ldots, n_1 - 2$, and $\eta_{b+n+1} \in [0, \eta_{b+n+1}]$;

for $h - j \in K_2 : z_{a-j} \in [1, \frac{\eta_{b-r}}{\rho}, 1)$,

$$\eta_{b-r} \in [0, 1), j = 0, 1, \ldots, n_2 - 2$$

and $\eta_{b+n+1} \in [0, \eta_{b+n+1}]$. \} (12)

The partition in fact corresponds to the stages of the two event lifetimes at the current and the previous transition times (up to the starting points of the two events). $u_i \in D_i$ implies that event $i$ enters stage 1 of the current lifetime at $j - n_i + 1$, and it enters the next stage at instants in $T_i$ (which is determined by $K_i$), $i = 1, 2$.

Thus, we can use the sequence of stages to characterise $x$. More precisely, if $u_i \in D_i$ shown in (12), we set $v_{(i)} = (y_{(i)}^{(1)}, y_{(i)}^{(2)}, \ldots, y_{(i)}^{(n_1)} + 1)$, where $y_{(i)}^{(j)}$ is the event of stage $i$ at $t_j$, $y_{(i)}^{(j)} = y_{(i)}^{(j)} + 1$ if $j \in T_i$, and $y_{(i)}^{(j)} = y_{(i)}^{(j)}$ if $j \not\in T_i$. It is easy to check that, given $v_{(i)}$, knowing $v_{(i+1)}$ does not help to determine $u_i$. That is, (5) and (6) hold for the partition in (12).

Therefore, $\mathbf{V}$ is a Markov chain.

For point processes with more than two variables, $V_{(i)}$ is a vector containing the sequences of stages for every event lifetime. The following theorem summarises the results.

Theorem 1. If the interarrival times of all the events in a point process have bounded hazard rates, then there exists an embedded Poisson process and an embedded Markov chain, which is defined on the Poisson process, with a countable state that represents the stages of the event lifetimes.

A more general case for the multivariable point processes is that the interarrival time may depend on the system state. The above results still hold in this case. We illustrate this by an example.

Example 2. The M/G/1 queue. We study an M/G/1 queue having a Poisson arrival with rate $\lambda_1$ and a service time distribution function $F(r)$ whose hazard rate has an upper bound $\lambda_2$. The system (physical) state is the number of customers in the queue, denoted as $N(t)$.

The arrival and departure times form a two-variable point process. The only distinction of the M/G/1 queue and the two-variable point process discussed above is that a departure cannot occur during an idle period (i.e., when $N(t) = 0$). This simply says that when $N(t) = 0$, the interdeparture time has a distribution different from $F(r)$. That is, the interdeparture distribution depends on the physical state.

References


