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An algebraic principle for blind separation of white non-Gaussian sources

Jie Zhu[a,*], Xi-Ren Cao[b], Zhi Ding[c]

[a]*Center for Signal Processing, S2-B4b-05, Nanyang Technological University, Singapore 639798, Singapore*
[b]*Department of Electrical & Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong*
[c]*Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242-1595, USA*

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Abstract

An algebraic principle for blind source separation is presented in this paper. This separation principle identifies a (smaller) set of equations whose solutions can blindly extract non-Gaussian signals. The concept of "Mth-order uncorrelatedness" is introduced and it is proven that for Mth-order uncorrelated source signals, signals with nonzero kth-order cumulant ($2 < k \leqslant M$) can always be extracted by setting a small set of kth-order cross-cumulants of output signals to zero. The set of kth-order cross-cumulants specified here is a sub-set of those used by other existing methods. The relationship between the algebraic principle and several existing algorithms is presented. The contributions of this principle are the reduction of the number of cross-cumulants used and the flexibility it affords in designing algorithms for blind source separation. © 1999 Published by Elsevier Science B.V. All rights reserved.

Zusammenfassung

In diesem Artikel wird ein algebraisches Prinzip zur blinden Separation von Quellen vorgestellt. Dieses Prinzip identifiziert eine (kleinere) Menge von Gleichungen, deren Lösungen Signale blind extrahieren können, die nicht normalverteilt sind. Das Konzept der "Unkorreliertheit M-ter Ordnung" wird eingeführt und es wird gezeigt, dass für Quellensignale, die bis zur Ordnung M unkorreliert sind, Signale mit von Null verschiedenen Kumulanten k-ter Ordnung ($2 < k < M$) immer extrahiert werden können, falls eine kleine Menge der den Ausgabesignale abgeleiteten Kreuzkumulanten k-ter Ordnung zu Null gesetzt wird. Die Menge der hier spezifizierten Kreuzkumulanten k-ter Ordnung ist eine Untermenge der Menge von Kreuzkumulanten, die von anderen existierenden Methoden verwendet wird. Die Beiträge dieses Prinzips sind die geringe Anzahl der zu verwendenden Kreuzkumulanten und die Flexibilität, welche zum Design von Algorithmen für die blinde Separation von Quellen bereitgestellt wird. © 1999 Published by Elsevier Science B.V. All rights reserved.

Résumé

Nous présentons dans cet article un principe algébrique pour la séparation aveugle de sources. Ce principe de séparation identifie un ensemble (plus petit) d'équations dont les solutions peuvent extraire en aveugle des signaux non-gaussiens. Le concept de "décorrélation d'ordre M" est introduit et il est prouvé que pour des signaux décorrélés

* Corresponding author. Tel.: + 65-790-6973; fax: + 65-791-0128; e-mail: ejzhu@ntu.edu.sg

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d'ordre M les signaux ayant des cumulants d'ordre k non nuls ($2 < k \leqslant M$) peuvent toujours être extraits en mettant un petit ensemble de cumulants croisés d'ordre k des signals de sortie à zéro. L'ensemble des cumulants croisés d'ordre k spécifié ici est un sous-ensemble de ceux utilisés par d'autres méthodes existantes. La relation entre le principe algébrique et plusieurs algorithmes existants est présentée. Les contributions de ce principe sont la réduction du nombre de cumulants croisés utilisés et la flexibilité qu'il apporte dans la conception d'algorithmes pour la séparation de sources aveugle.

Keywords: Blind signal separation; Independent component analysis; Higher-order statistics; Principle component analysis

1. Introduction

In the past decade, many research efforts have been devoted to the blind source separation (BSS) [3,8,9,15,18,21,27,30] or equivalently, independent component analysis (ICA) [13,17,20,24] The BSS problem can be described as follows. Denote the number of sources as n, which is assumed to be known. Let $\boldsymbol{s}(t) = [s_1(t), s_2(t), \ldots, s_n(t)]'$ be the vector of unknown signals or "source" signals, where the superscript $'$ represents *transpose*. Then a known mixture of the source signals is modeled by

$$\boldsymbol{o}(t) = \boldsymbol{A}\boldsymbol{s}(t), \tag{1}$$

where $\boldsymbol{o}(t) = [o_1(t), o_2(t), \ldots, o_m(t)]'$ and $\boldsymbol{A} = [a_{i,j}] \in R^{m \times n}$ is the unknown *mixing matrix* representing the unknown physical environment in which signals are transmitted and mixed. In order to be solvable, the BSS problem requires that the mixing matrix $\boldsymbol{A}$ must be of non-singular or full column rank when $m = n$ or $m > n$, respectively. Without loss of generality, we assume that $\boldsymbol{A}$ is an $n \times n$ square matrix with non-singularity hereafter.

The objective of BSS is to recover source signal $\boldsymbol{s}(t)$ by using the observed signal $\boldsymbol{o}(t)$ and the assumption of mutual independence among source signals. It exploits only the information carried by the observed signals themselves. For implementation, the source signals are assumed to be ergodic, or equivalently the observed signals are ergodic. It is worth emphasizing that the BSS problem in this strict sense only relies on the assumption of mutual independence among source signals. Other techniques, such as Cyclostationary Signal Separation [2] and AMUSE-type algorithms [5,28] where either cyclostationary source signals or colored source signals with different spectral shapes are required, are less general and beyond the scope of our consideration. When the matrix $\boldsymbol{A}$ is non-singular (more general cases are in [8]), there exists a non-singular $n \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{B}$ extracts all source signals via

$$\boldsymbol{e}(t) = \boldsymbol{B}\boldsymbol{o}(t) = \boldsymbol{B}\boldsymbol{A}\boldsymbol{s}(t). \tag{2}$$

We refer to $\boldsymbol{B}$ as *separating matrix*. Because the relative amplitudes of source signals in $\boldsymbol{s}(t)$ and columns of matrix $\boldsymbol{A}$ are unknown, the separating matrix has a scaling freedom on each of its rows. In addition, the rows of $\boldsymbol{B}$ or the output signals can be permutated after separation. Hence, if $\boldsymbol{B}$ is a separating matrix then $\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{B}$ is also a separating matrix where $\boldsymbol{\Lambda}$ is any invertible diagonal matrix and $\boldsymbol{P}$ is a permutation matrix. The scaling and permutation properties are therefore two inherent indeterminacies of the underlying BSS problem.

Many approaches of the BSS problem have been proposed in the literature since the first heuristic but effective algorithm proposed by Jutten and Herault [21]. These approaches can be classified as adaptive (neural network) methods [4,7,11,14,19, 22,23] and algebraic methods [12,17,26,29]. Adaptive methods typically rely on a suitable cost function in terms of higher-order statistics of the output signals for minimization. It updates the separating matrix $\boldsymbol{B}$ with each arrival of mixed signal samples. The algebraic method, on the other hand, first computes the separating matrix $\boldsymbol{B}$ from an ensemble of mixed samples and recovers the source signals by (2). In this paper we are interested in the algebraic method and determine the separating matrix $\boldsymbol{B}$ by solving a set of equations involving the statistics of the output signals.

The first algebraic method for BSS is the *Fourth-Order Blind Identification* (FOBI) algorithm proposed by Cardoso [9] and later extended by Tong et al. [27] to noisy systems. This method is in fact based on the decomposition of a special matrix about the fourth-order cumulants of the estimated signal $\boldsymbol{e}(t)$. When each source signal has a unique normalized *kurtosis* (i.e. the ratio of fourth-order cumulant to the square of signal energy), the matrix decomposition is unique in the sense of BSS, which means that the source signals can be separated. To eliminate additional conditions on the source kurtosis, Cardoso [12] later developed another algorithm named JADE. JADE is designed on the *Joint Approximate Diagonalisation* of all 4th order cumulant matrices of the estimated signals and thereby is more robust than FOBI. Comon [17] proposed a cost function to determine the pairwise independence among the estimated signals. The algorithm is based on only fourth-order statistics and is hence similar in spirit to JADE. An overview of the algebraic methods was recently presented in [10]. In contrast to adaptive methods, algebraic methods have no convergent concerns and can be simplified by using eigen-decomposition of higher-order cumulant matrices and other algebraic techniques. However, the computation of higher-order cumulant matrices in the algebraic method can be complex, especially for those non-Gaussian signals with zero cumulant up to some large order.

In this paper, we propose an algebraic principle for blind source separation. The principle establishes that a set of source signal can be separated by forcing some higher-order cross-cumulants to be zero even if the source signals are not mutually independent. We first introduce the concept of "Mth-order uncorrelatedness" to describe a statistical relationship among a set of source signals. Then we prove that if the source signals are mutually Mth-order uncorrelated ($M > 2$), a source signal with nonzero kth-order cumulant can always be separated out by setting a small set of kth-order cross-cumulants of the estimated signals to zero ($k \leqslant M$). The number of the cross-cumulants is $n(n-1)$ for an odd number k and $(n+1)(n-1)$ for an even number k. Finally, we present the relationship between this algebraic principle and some other existing algorithms.

The paper is organized as follows. Section 2 reviews the basic statistic information useful for algebraic BSS methods. In Section 3, Mth-order uncorrelated source signals are defined and the set of higher-order cross-cumulants for source signal separation are established. Section 4 describes the relationship between our algebraic principle and other existing methods.

2. Statistics in blind source separation

The task of BSS is to find the separating matrix $\boldsymbol{B}$. From the algebraic point of view, it involves to build and solve a set of equations about $\boldsymbol{B}$. $\boldsymbol{B}$ is an $n \times n$ matrix thus has n^2 unknown entries. Because of the inherent indeterminacy of the scale on each row of $\boldsymbol{B}$, there are totally $n^2 - n$ unknowns in $\boldsymbol{B}$. Thus, the algebraic equation set used to determine $\boldsymbol{B}$ must contains at least $n^2 - n$ independent equations. If $\boldsymbol{e}(t)$ is a copy of $\boldsymbol{s}(t)$ up to their scales and order permutation, then the components of $\boldsymbol{e}(t)$ must be uncorrelated signals upon successful separation. Hence, by

$$E\{\boldsymbol{e}(t)\boldsymbol{e}(t)^{\dagger}\} = \boldsymbol{I}_n,$$

we can generate $n(n-1)/2$ independent equations, where the superscript $\dagger$ stands for *complex conjugate transpose* and $\boldsymbol{I}_n$ is an $n \times n$ identity matrix. To determine $\boldsymbol{B}$ fully, a straightforward idea is to collect additional $n(n-1)/2$ equations from higher-order statistics of the output signals. With this basic idea in mind, almost all of the algebraic algorithms can be divided into two stages: "whitening" and "rotating".

The whitening stage is to find a matrix $\boldsymbol{W}$ by which the components of $\boldsymbol{z}(t) = \boldsymbol{W}\boldsymbol{o}(t)$ are uncorrelated. This is a typical *principle component analysis* (PCA) problem and can be solved by many well-designed algorithms. The rotation stage is to find an orthogonal matrix $\boldsymbol{U}$ to finally separate out the source signals: $\boldsymbol{e}(t) = \boldsymbol{U}\boldsymbol{z}(t)$. This algebraic procedure can be illustrated in Fig. 1.

The final separating matrix is given by $\boldsymbol{B} = \boldsymbol{U}\boldsymbol{W}$. Obviously, the key point in this procedure is to

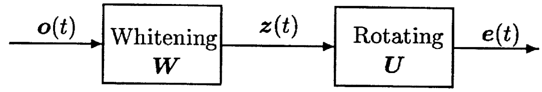

Fig. 1. Block diagram of the algebraic method.

build at least $n(n-1)/2$ linear independent equations based on the higher-order statistics so that the whitened signal $\boldsymbol{z}(t)$ can be rotated to a suitable point in the signal space.

One may notice that whitening the observed signals can always be achieved by PCA. However, rotating the whitened signals requires sufficient message from their higher-order statistics. As the higher-order cumulants of a Gaussian source are always zero, it is apparent that Gaussian sources cannot be separated. For non-Gaussian source signals, Tong et al. [26] proved that the BSS can always be realized. Tong et al. concluded that if a source signal $s_i(t)$ has nonzero kth-order cumulant, then the ith column of the mixing matrix $\boldsymbol{A}$ can be identified by solving a set of equations related to the kth-order statistics of source signals. The set of equations was named as the *kth-order identification equations*. However, the size of the kth identification equations increases exponentially with the order of cumulants and the number of sources. When the order k gets large, it becomes a formidable to deal with all the kth-order identification equations. This motivates us in this paper to investigate whether or not all the kth-order identification equations need to be involved.

Note that in the BSS model the source signals are often assumed to be mutually independent. This assumption implies that any cross-cumulant of source signals must be zero and is quite strong. Because signals in practical systems can be complex, source signals cannot and need not be tested for "complete" mutual independence. In fact, most existing algebraic algorithms, such as the one developed in [26], were developed for sources with nonzero cumulants of order k. Thus, they can be applied to separation of source signals mutually uncorrelated up to order k, no matter whether the sources are mutually independent or not.

In the next section, we will first introduce a concept to represent a set of source signals that are uncorrelated but not mutually independent. Their separability does not require the testing of their mutual independence. We then study the identification equations to determine the rotation $\boldsymbol{U}$ and prove that instead of solving all identification equations, a small part of identification equations are sufficient for determining $\boldsymbol{U}$.

3. Mth-order uncorrelated sources and their separations

For notational simplicity, we may drop the time index t from signals when there is no possibility of confusion, i.e. $\boldsymbol{s}(t) \equiv \boldsymbol{s}$, $\boldsymbol{o}(t) \equiv \boldsymbol{o}$, $\boldsymbol{e}(t) \equiv \boldsymbol{e}$.

3.1. Mth-order uncorrelated sources

Resorting to the cumulant concept, we will define a class of signals as *Mth-order uncorrelated sources*. M is a positive integer. The constraints for Mth-order uncorrelated sources are stronger than that for (second order) uncorrelated sources, but weaker than for mutually independent sources.

Define P_s and P_{s_i} as the probability density functions of $\boldsymbol{s}$ and s_i, respectively. Let $\phi_s(\boldsymbol{t})$ be the joint *second characteristic function* (SCF) of the source signal $\boldsymbol{s}$ where $\boldsymbol{t} = [t_1, t_2, \dots, t_n]'$, and $\phi_{s_i}(t_i)$ be the marginal SCF of the ith source signal

$$\phi_s(\boldsymbol{t}) = \log\left(\int_{-\infty}^{+\infty} \exp(-\mathrm{j}\boldsymbol{t}'\boldsymbol{s}) P_s \,\mathrm{d}\boldsymbol{s}\right),$$

$$\phi_{s_i}(t_i) = \log\left(\int_{-\infty}^{+\infty} \exp(-\mathrm{j}t_i s_i) P_{s_i} \,\mathrm{d}s_i\right).$$

It is well known that the source signals are mutually independent if and only if the following equality holds:

$$\phi_s(\boldsymbol{t}) = \sum_{i=1}^{n} \phi_{s_i}(t_i). \tag{3}$$

Denote the Mth-order cumulant of $\boldsymbol{s}$ as

$$C_s(p_1,p_2,\dots,p_n) = (-\mathrm{j})^M \left|\frac{\partial^M \phi_s(\boldsymbol{t})}{\partial t_1^{p_1} \cdots \partial t_n^{p_n}}\right|_{\boldsymbol{t}=0},$$

$$M = p_1 + \cdots + p_n, \tag{4}$$

where p_i are non-negative integers. Then from Taylor expansion, (3) is equivalent to

$$C_s(p_1, p_2, \dots, p_n) = 0, \tag{5}$$

for all p_i, $i = 1,2,\dots,n$, and at least two of p_i's are nonzero. The equation set of (5) completely describes the mutual independence of a set of source signals in terms of the higher-order statistics. It indicates that the sources are mutually independent if and only if their cross-cumulants are all zeroes.

Similarly, we define the concept of *Mth-order uncorrelated* sources using the cumulant approach.

Consider a set of random variables which are not mutually independent. These random variables must have some nonzero cross-cumulants. However, if they are uncorrelated, i.e. their second cross-cumulants are all zeros, then these random variables can be seen as being "more" mutually independent than correlated random variables. Similarly, we consider random variables with more zero cross-cumulants as "more" mutually independent. Consequently, we can use the following definition to characterize the level of independence for uncorrelated signals.

Definition. Let p_i, for $i = 1,2,\dots,n$, be nonnegative integers, $p_1 + p_2 + \cdots + p_n \leqslant M$ and at least two of these integers are nonzero. A set of sources, s_i, $i = 1,2,\dots,n$, is said to be mutually Mth-order uncorrelated if $C_s(p_1, p_2, \dots, p_n) = 0$.

Obviously, the fact that a set of sources are mutually independent is equivalent to that they are mutually Mth-order uncorrelated for any positive integer M. If a set of sources is mutually Mth-order uncorrelated, then its components are mutually mth-order uncorrelated for any $0 < m \leqslant M$. We can now prove the following theorem.

Theorem 1. *Let* $\boldsymbol{e} = \boldsymbol{Ds}$ *and* $\boldsymbol{D} = [d_{ij}]_{i,j=1,\dots,n}$. *If all* s_k, $k = 1,2,\dots,n$, *are mutually Mth-order uncorrelated and their cumulants up to Mth-order exist, then*

$$C_{e_i e_j}(p,q) = \sum_{k=1}^{n} d_{i,k}^{p} d_{j,k}^{q} C_{s_k}^{p+q},$$

where p and q are any non-negative integers satisfying $p + q \leqslant M$, *and* $C_{s_k}^{p+q}$ *is the* $(p+q)$*th-order cumulant of the kth source signal.*

By using the property of cumulants [6] and the definition given in the above, the proof of this theorem is not difficult and for convenience is presented in Appendix A. Theorem 1 states that the mth-order (cross) cumulant of a sum of mutually Mth-order uncorrelated source signals is equal to the sum of the individual mth-order (cross) cumulants of their source signals, $m \leqslant M$. This conclusion is similar to that obtained when the source signals are mutually independent. In fact, if a set of source signals are mutually Mth-order uncorrelated, then the statistical exhibition of the set of source signals is equal to that of mutually independent sources up to Mth-order statistics.

3.2. Separation of sub-Gaussian and super-Gaussian sources

Note that in Theorem 1 the cumulants of source signals can be any nonzero numbers. However, in some applications, the cumulants of a certain order of all source signals may be either negative or positive. For example, speech signals typically have positive fourth-order cumulants. Source signals that have negative fourth-order cumulants are often called "sub-Gaussian" signals and those source signals with positive fourth-order cumulants are called "super-Gaussian" signals. In these environments, the BSS might be simplified by the following corollary.

Corollary 1. *Let k, p be two even integers and* $2 \leqslant p < k$, *and* $\boldsymbol{e} = \boldsymbol{Bo}$ *where* $\boldsymbol{B}$ *is non-singular. If source signals are mutually kth-order uncorrelated and all their kth-order cumulants are positive* (*negative*), *then source signals can be separated in* $\boldsymbol{e}$ *by minimizing* (*maximizing*)

$$f(\boldsymbol{B}) = \sum_{i=1}^{n} \sum_{j \neq i}^{n} C_{e_i e_j}(k-p,p), \tag{6}$$

where n is the number of sources.

Proof. From Eq. (1), $\boldsymbol{e} = \boldsymbol{Bo} = \boldsymbol{BAs}$. Let $\boldsymbol{D} = \boldsymbol{BA}$. Without loss of generality, let all kth-order cumulants be positive. According to Theorem 1, for all $i, j = 1,2,\dots,n$,

$$C_{e_i e_j}(k-p,\ p) \geqslant 0, \quad i \neq j. \tag{7}$$

The minimum of $C_{e_i e_j}(k-p,p)$ leads to either $d_{i,k} = 0$ or $d_{j,k} = 0$ or $d_{i,k} = d_{j,k} = 0$ where $d_{i,j}$ is the (i,j)th component of $\boldsymbol{D}$. Since minimizing $f(\boldsymbol{B})$ is equivalent to minimizing $C_{e_i e_j}(k-p,p)$ respectively for all $i \neq j$, the minimum of $f(\boldsymbol{B})$ will result in no more than one nonzero $d_{i,k}$ for any k. As $\boldsymbol{B}$ is nonsingular and $\boldsymbol{D}=\boldsymbol{BA}$, $\boldsymbol{D}$ is nonsingular as well. Consequently, $\boldsymbol{D}$ does not have any zero column thereby $\boldsymbol{D}$ is a permutation matrix. □

For those applications where all cumulants of sources have the same sign, Corollary 1 suggests a simple way to separate the source signals. Very often, source signals emitted by some similar physical procedures, such as the baseband signals from the same constellation in wireless communication systems, generally have quite similar statistical properties. This corollary becomes very useful in these environments.

However, if the sources have different probability distribution, then some of their kth-order cumulants might be positive and the others negative. Such cases are investigated next.

3.3. Separations of sources with nonzero odd-order cumulants

When the distribution of a source signal is not symmetric, the source signal perhaps has a nonzero cumulant of odd order. For such source signals, we have the following theorem.

Theorem 2. *Let k be an odd integer and $k>2$. If the source signals are mutually Mth-order uncorrelated where $k \leqslant M$, and $\boldsymbol{e}=\boldsymbol{Bo}$ so that*

$$\begin{aligned} &C_{e_ie_j}(1,1)=0, \\ &C_{e_ie_j}(k-1,1)=0, \end{aligned} \quad \text{for } i,j=1,2,\dots,n,\ i\neq j, \tag{8}$$

then all source signals with nonzero kth-order cumulant are separated in $\boldsymbol{e}$.

Theorem 2 asserts that if the source signals are mutually Mth-order uncorrelated , then by making the estimated signals white and some kth-order cross-cumulants equal to zero simultaneously where k is an odd integer and $2<k\leqslant M$, those source signals with nonzero kth-order cumulants can be separated. To prove Theorem 2, we need to present a lemma.

Lemma 1. *Let k be an odd integer and $k>2$. Suppose that an $n\times n$ nonsingular matrix $\boldsymbol{D}=[d_{i,j}]_{i,j=1}^{n}$ is the solution to*

$$\begin{aligned} &\sum_{l=1}^{n} d_{i,l}d_{j,l}=0, \\ &\sum_{l=1}^{n} d_{i,l}^{k-1}d_{j,l}\beta_l=0, \end{aligned} \quad i,j=1,2,\dots,n,\ i\neq j. \tag{9}$$

If $\beta_m\neq 0$, $1\leqslant m\leqslant n$, then the mth column of $\boldsymbol{D}$ has only one nonzero entry and this nonzero entry is in a row in which all the other entries are zeros. Particularly, if there is at most one of β_l, $l=1,2,\dots,n$, being zero, then $\boldsymbol{D}$ is a permutation matrix.

The proof of Lemma 1 is presented in Appendix A. Recall that in Eq. (1) if $\boldsymbol{BA}$ is a permutation matrix then the estimated signals are the source signals without considering their ordering and scaling. Lemma 1 explains the conditions under which a non-singular matrix must be a permutation matrix. Letting $\boldsymbol{D}=\boldsymbol{BA}$, Theorem 1 allows the cross cumulants of $\boldsymbol{e}$ be written in the equation form of (9). Thus, Theorem 2 is immediate from Lemma 1.

3.4. Separations of sources with nonzero even-order cumulants

In most applications, the probability distributions of source signals are symmetric or nearly symmetric. The odd-order cumulants of sources will approach to zero. The separation of these source signals is therefore built on their even-order statistics. For a source signal with nonzero even-order cumulant, we have the theorem below.

Theorem 3. *Let k and p be two even integers and $2\leqslant p<k$. If the source signals are mutually Mth-order uncorrelated where $k\leqslant M$, and $\boldsymbol{e}=\boldsymbol{Bo}$ so that*

$$\begin{aligned} &C_{e_ie_j}(1,1)=0, \\ &C_{e_ie_j}(k-1,1)=0, \\ &C_{e_re_j}(k-p,p)=0, \end{aligned}$$

$$\text{for } i,j=1,2,\dots,n,\ i\neq j,\ j\neq r, \tag{10}$$

where $1\leqslant r\leqslant n$, then a source signal with nonzero kth-order cumulant is separated in $\boldsymbol{e}$.

Similar to Theorem 2, Theorem 3 provides a set of equations for finding the separating matrix $\boldsymbol{B}$ where the source signals have nonzero even-order cumulants. The number of equations is now $(3n+2)(n-1)/2$. Theorem 3 can be proven by resorting to the following lemma.

Lemma 2. *Let k and p be two even integers and $2 \leqslant p < k$. Suppose that an $n \times n$ nonsingular matrix $\boldsymbol{D} = [d_{i,j}]_{i,j=1}^{n}$ is the solution to*

$$\sum_{l=1}^{n} d_{i,l} d_{j,l} = 0,$$
$$\sum_{l=1}^{n} d_{i,l}^{k-1} d_{j,l} \beta_l = 0,$$
$$\sum_{l=1}^{n} d_{j,l}^{k-p} d_{r,l}^{p} \beta_l = 0,$$
$$i,j = 1,2,\dots,n, \quad i \neq j, \quad j \neq r, \tag{11}$$

where r is a positive integer satisfying $1 \leqslant r \leqslant n$. If $\beta_m \neq 0$, $1 \leqslant m \leqslant n$, then the mth column of $\boldsymbol{D}$ has only one nonzero entry and this entry is the only nonzero entry in its row. In particular, if there is at most one zero β_l, $l = 1,2,\dots,n$, $\boldsymbol{D}$ is a permutation matrix.

Proof. See Appendix A. □

From Lemma 2 and Theorem 1, Theorem 3 is straightforward.

4. Relationship to other algebraic methods

In the last section, we have presented two theorems for blind separation of non-Gaussian source signals. For signals with nonzero odd-order cumulants, Theorem 2 suggests that $n(n-1)$ cross-cumulants of the outputs can be forced to zero in the rotation stage, whereas for signals with nonzero even-order cumulants, Theorem 3 suggests that $(n+1)(n-1)$ cross-cumulants be force to zero. In comparison with algorithms such as the one in [26], the number of equations involving kth-order cross-cumulants has been significantly reduced. When computational cost is crucial, the reduced number of equations makes BSS implementation efficient.

It is worth mentioning that Theorems 2 and 3 provide a general separation principle for non-Gaussian source signals. They are not limited to source signals with nonzero cumulants of a particular order. To develop a practical and robust algorithm, one can either directly solve the equation set in Theorem 2 or in Theorem 3 (which may not be a good choice), or apply some nonlinear optimization algorithms to exploit the eigen structure of matrices consisting of cross cumulants in Theorems 2 and 3. From this standpoint, Theorems 2 and 3 establish an algebraic principle for BSS. We will show later that some well-known algorithms for BSS, such as JADE proposed by Cardoso [12], can be seen as an implementation of the separation principle for the separation of source signals with nonzero fourth-order cumulants.

In order to illustrate that our equation sets are essential, we consider some well-known BSS methods and show that they are special implementations of our separation principle. The methods to be considered include Comon's method [17], Cardoso's JADE [12], and recent results by Nadal [24]. As the first two methods are based on the fourth-order statistics of source signals extended to complex signal applications, for the sake of clarity we denote e_i^* as the complex conjugate of e_i and restate Theorem 3 for $k = 4$ in complex signal applications as follows.

Corollary 2. *Let $\boldsymbol{e} = \boldsymbol{B}\boldsymbol{o}$. If the source signals are mutually fourth-order uncorrelated and*

$$C_{e_i e_j^*}(1,1) = 0,$$
$$C_{e_i e_i^* e_i e_j^*}(1,1,1,1) = 0,$$
$$C_{e_r e_j^*}(2,2) = 0,$$
$$\text{for } i,j = 1,2,\dots,n, \quad i \neq j, \quad j \neq r, \tag{12}$$

where r is a positive integer and $1 \leqslant r \leqslant n$, then a source signal with nonzero fourth-order cumulant is separated in $\boldsymbol{e}$.

4.1. Relationship to Comon's algorithm

In [17], Comon introduced Edgeworth expansion to approximate both joint and marginal PDFs of sources and proposed an ICA-based algorithm. Under the assumption that the source signals have nonzero fourth-order cumulants, this algorithm is capable of separating all source signals.

After whitening the observed signals, Comon developed a scheme for rotating the whitened signals by maximizing the criterion

$$f(\boldsymbol{U}) = \sum_{i=1}^{n} \|C_{e_i e_i^*}(2,2)\|^2$$

subject to the condition that $\boldsymbol{U}$ must be a unitary matrix. Denote the whitened signal by $\boldsymbol{z}$. Since the sum of all squared fourth-order (cross) cumulants of $\boldsymbol{e} = \boldsymbol{U}\boldsymbol{z}$ is a constant as long as $\boldsymbol{U}$ is kept unitary, the above criterion is in fact equivalent to minimize all of squared fourth-order cross cumulants, i.e.

$$\begin{aligned} &C_{e_i e_j^*}(1,1) = 0, \\ &C_{e_i e_j^* e_k e_l^*}(1,1,1,1) = 0 \end{aligned} \tag{13}$$

for all $i,j,k,l = 1,2,\dots,n$, excluding $i = j = k = l$. It is easy to see that the equation set (12) is a sub-set of (13) by letting $j = k = l$ and $i = j = r$, $k = l \neq r$. Therefore, the solution to (13) must satisfy (12).

4.2. Relation to JADE algorithm

Cardoso proposed a strategy for rotating the whitened signals with nonzero fourth-order cumulants in JADE [12]. The rotation is achieved by maximizing the criterion

$$f(\boldsymbol{U}) = \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \|C_{e_i e_i^* e_k e_l^*}(1,1,1,1)\|^2 \tag{14}$$

subject to the unitary constraint. This criterion is equivalent to forcing

$$\begin{aligned} &C_{e_i e_j^*}(1,1) = 0, \\ &C_{e_i e_j^* e_k e_l^*}(1,1,1,1) = 0, \\ &\quad \text{for } i,j,k,l = 1,2,\dots,n,\ i \neq j, \end{aligned} \tag{15}$$

By letting $i = k = l$ and $i = k$, $j = l = r$, the criterion gives the equation set (12). Again, (12) is a sub-set of (15). A solution of (15) must be the solution of (12).

4.3. Relation to Nadal's results

Recently, Nadal and Parga [24] proved some general results on BSS. For source signals with nonzero odd-order cumulants, Theorem 1 in [24] is equivalent to our Theorem 2. However, for the source signals with nonzero even-order cumulants, the authors suggested the following equation set:

$$\begin{aligned} &C_{e_i e_j}(1,1) = 0, \\ &C_{e_i e_j e_l}(k-m, m-1, 1) = 0, \\ &\quad \text{for } i,j,l = 1,2,\dots,n, \text{ but } i = j = l, \end{aligned} \tag{16}$$

where k and m are two positive integers, $2 \leqslant m < k$. If a source signal has nonzero kth-order cumulant, then the source signal is separated in $\boldsymbol{e}$. Although this equation set is flexible on m, it is indeed a huge equation set. The maximum number of equations is $n(n-1)/2 + (n^3 - n) = n(n+1.5)(n-1)$ for $k - m \neq m - 1 \neq 1$. However, the minimum number of equations is $n(n-1)/2 + (n^3 - n)/2 = n(n^2 + n - 2)/2$ when either $k - m$ or $m - 2$ is equal to 1. In fact, the equation set of Theorem 3 is also a subset of (16). To show this point, let us consider the cases of an odd m and an even m, respectively. If m is an odd integer, then $m - 1$ is an even integer equivalent to p in Theorem 3. The equation set (10) therefore can be obtained from (16) by letting $i = j$ or $i = k$ and $j = r$. On the other hand, if m is an even integer, the equation set in Theorem 3 consists of those equations of (16) with $i = j$ or $j = k = r$.

In summary, we have established a set of cross-cumulant equations for BSS. The number of cross-cumulants of outputs is fixed at $n(n-1)$ for the separation of source signals with nonzero odd-order cumulants, and is equal to the result in [24]. On the other hand, the number of cross-cumulants for the separation of source signals with nonzero even-order cumulant is $(n+1)(n-1)$. This number is a little larger than that for nonzero odd-order cumulant sources but much smaller than those in the literature. For clarity, we list the number of higher even-order equations used in each method in Table 1. Here, k is an even integer and Comon's method and JADE are limited to the fourth order. Table 1 clearly indicates that the equation set suggested by our principle is indeed the smallest. Note that the number of their second-order equations is not included and is always $n(n-1)/2$. For odd-order HOS, both Nadal's results and our results lead to the same equation set whose number is $n(n-1)$.

Table 1
The number of even-order equations

Comon's approach	JADE algorithm	Nadal's approach	Our approach
$n^4 - n$	n^3	$n(n-1)(n+1)/2$	$(n+1)(n-1)$

It should be emphasized that although a smaller set of sufficient equations were proposed for blind source separation, it is not recommended that only the set of equations be directly solved. In fact, any algorithm, such as those mentioned in this section, is capable of source separation as long as its solution satisfies these equations. In practical implementation of BSS algorithms, the nonlinearity of the equation set should be taken into account. Since the order of the equation set is larger than 2, cost functions involving only this set of equation may not be convex or unimodal. On the other hand, additional equations may be included to form optimization algorithms that are unimodal. Thus, it is computationally more efficient for a unimodal BSS algorithm to include additional equations in its solution. Moreover, BSS algorithms that satisfy more statistical equations may also be numerically more robust. One such an example is the JADE algorithm. Therefore, our contribution in this paper is an algebraic principle, instead of algorithms, for blind separation of non-Gaussian source signals. One can utilize this principle to design specific BSS algorithms.

5. Conclusions

In this paper we established an algebraic principle for blind separation of non-Gaussian source signals. The source signals can be either white or color. The principle suggested two cross-cumulant sets: one is for the separation of those source signals with nonzero odd-order cumulants, and the other is for with nonzero even-order cumulants. The source signals may not be mutually independent but should be mutually Mth-order uncorrelated. We illustrated that these two cross cumulant equation sets are the sub-set of those used in some existing algebraic methods. Any method involving these two equation sets is always able to separate the source signals with nonzero corresponding order cumulants.

For further reading see [1,16,25].

Appendix A

Proof of Theorem 1. Let $e_i = \sum_{k=1}^{n} d_{i,k} s_k$ and $e_j = \sum_{k=1}^{n} d_{j,k} s_k$. Then by the algebraic property of cumulants ([6, pp. 174–175]), we have

$$C_{e_i e_j}(p,q) = \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_q=1}^{n} d_{i,i_1} \cdots d_{i,i_p} d_{j,j_1} \cdots d_{j,j_q} C_s(m_1,m_2,\ldots,m_n),$$

where m_k for $k = 1,2,\ldots,n$ is the number of $i_1,\ldots,i_p$, $j_1,\ldots,j_q$ whose values are all equal to k. Since the components of $\boldsymbol{s}$ are mutually Mth-order uncorrelated , their Mth-order cross-cumulants are all zeros. We then obtain

$$\begin{aligned} C_{e_i e_j}(p,q) &= C_s(p+q,0,\ldots,0) d_{i,1}^p d_{j,1}^q \\ &\quad + C_s(0,p+q,\ldots,0) d_{i,2}^p d_{j,2}^q \\ &\quad + C_s(0,0,\ldots,p+q) d_{i,n}^p d_{j,n}^q, \end{aligned}$$

which is the conclusion. □

Proof of Lemmas 1 and 2. Since $\boldsymbol{D}$ is non-singular, in each column of $\boldsymbol{D}$ there is at least one entry whose cofactor and itself are both nonzero. Consider the mth column of $\boldsymbol{D}$. Without loss of generality, we assume that $d_{1,m} = 1$ and its cofactor is nonzero. Let $i = 1$ and $j = 2,3,\ldots,n$. The first equation of (9) yields

$$\begin{bmatrix} d_{2,1} & \cdots & d_{2,m-1} & d_{2,m+1} & \cdots & d_{2,n} \\ d_{3,1} & \cdots & d_{3,m-1} & d_{3,m+1} & \cdots & d_{3,n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ d_{n,1} & \cdots & d_{n,m-1} & d_{n,m+1} & \cdots & d_{n,n} \end{bmatrix} \times \begin{bmatrix} d_{1,1} \\ \vdots \\ d_{1,m-1} \\ d_{1,m+1} \\ \vdots \\ d_{1,n} \end{bmatrix} = - \begin{bmatrix} d_{2,m} \\ d_{3,m} \\ \vdots \\ d_{n,m} \end{bmatrix}. \tag{17}$$

From the second equation of (9), we have

$$\begin{bmatrix} d_{2,1} & \cdots & d_{2,m-1} & d_{2,m+1} & \cdots & d_{2,n} \\ d_{3,1} & \cdots & d_{3,m-1} & d_{3,m+1} & \cdots & d_{3,n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ d_{n,1} & \cdots & d_{n,m-1} & d_{n,m+1} & \cdots & d_{n,n} \end{bmatrix}$$

$$\times \begin{bmatrix} d_{1,1}^{k-1}\beta'_1 \\ \vdots \\ d_{1,m-1}^{k-1}\beta'_{m-1} \\ d_{1,m+1}^{k-1}\beta'_{m+1} \\ \vdots \\ d_{1,n}^{k-1}\beta'_n \end{bmatrix} = -\begin{bmatrix} d_{2,m} \\ d_{3,m} \\ \vdots \\ d_{n,m} \end{bmatrix}, \tag{18}$$

where $\beta'_l = \beta_l/\beta_m$, $1 \leqslant l \leqslant n$, $l \neq m$. Comparing (17) with (18), we obtain $d_{1,l}^{k-1}\beta'_l = d_{1,l}$. Thus,

$$\text{either} \quad d_{1,l} = 0, \quad \text{or} \quad d_{1,l}^{k-2}\beta'_l = 1$$
$$\text{for all } 1 \leqslant l \leqslant n,\ l \neq m. \tag{19}$$

From (19), we can prove in the following that all $d_{1,l}$, $1 \leqslant l \leqslant n$, $l \neq m$, must be zero. Under the conditions of Lemma 1, if for some l, $d_{1,l}^{k-2}\beta'_l = 1$, then applying the second equation of (9) gives

$$\sum_{l=1}^{n} d_{j,l}^{k-1}/d_{1,l}^{k-3} = 0.$$

Because k is an odd integer, $d_{j,l} = 0$ for all $j,l = 1,2,\dots,n$. Under the conditions of Lemma 2, $k > 3$ is an even integer, by the third equation we have

$$\sum_{l=1}^{n} d_{r,l}^{p}/d_{1,k}^{p-2} = 0, \qquad r \neq 1,$$
$$\sum_{l=1}^{n} d_{i,l}^{k-p}/d_{r,l}^{k-p-2} = 0, \quad r = 1$$

which results in a singular $\boldsymbol{D}$. Therefore, $d_{1,l} = 0$ for $l = 1,2,\dots,n$, $l \neq m$.

To prove $d_{l,m} = 0$ for $l = 2,3,\dots,n$, we let $d_{1,l} = 0$ in (17). It is straightforward to see that $\boldsymbol{D}$ is a generalized permutation matrix when there exists at most one β_l being zero. $\square$

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