Ensemble-Average Importance Sampling of Markov Processes

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In this article, we introduce a new performance estimation method, called ensemble-average importance sampling (EAIS) of Markov processes. With EAIS, we can estimate, by observing a single sample path of a Markov process (which may be obtained by a single simulation run), the steady-state performance measure of any other Markov process having the same state space. In EAIS, the K-step state-transition matrix, instead of the time average of the performance function, is estimated by using ensemble averages. The variance of the estimate is much smaller than that of the straightforward application of the importance sampling method to the time average. EAIS does not resort to the regenerative property. The method is very efficient when used in parallel computers to simulate the performance of many systems simultaneously. Numerical examples for Markov chains and queueing networks are presented.

1 INTRODUCTION

Importance sampling is a technique commonly used to reduce the variance in simulation. The idea is simple. Let \( h(x) \) be the probability density function of a random variable \( x \), \( f(x) \) a performance function, and \( E[f(x)] = \int f(x) h(x) \, dx \) its mean value. Suppose \( g(x) \) is the probability density function of another random variable \( y \). One can write

\[
E[f(x)] = \int f(x) h(x) \, dx = \int \left\{ f(x) \frac{h(x)}{g(x)} \right\} g(x) \, dx.
\]

Based on this equation, the mean value \( E[f(x)] \) can be estimated by using a sequence of random samples of \( y_1, y_2, \ldots, y_N \), which is generated from the density function \( g(x) \), according to the following equation,

\[
E[f(x)] = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left\{ f(y_i) \frac{h(y_i)}{g(y_i)} \right\}.
\]  

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By choosing appropriate $g(x)$'s for particular $h(x)$'s, different techniques have been
developed to reduce the variance in estimating $E[f(x)]$ (see, e.g., (1)).

Equation (1) can be explained from another point of view: One can obtain the
mean performance $E[f(x)]$ for all random variables by generating a sequence of
random samples of one random variable $y$. This has the following interpretation: If
the probability density functions are known, then in simulation the randomness of
all these random variables can be determined by any one of the random variables
and the rest of the calculations are all deterministic.

The preceding idea is more significant when applied to estimating the perform-
ance of large complex systems (2). In this case, applying the method enables us
to obtain the mean performance of a system by observing a sample path of another
system that has a similar structure. Note that in Equation (1) the explicit forms of
the probability density functions $h(x)$ and $g(x)$ are not needed as long as the ratio
$h(x)/g(x)$ is known. For many systems, $h(x)$ and $g(x)$ are known to us, but to obtain
their ratio may be easier. As discussed in Section 2, however, if one applies the idea
directly to the estimates of the steady-state performance of a stochastic system, one
usually encounters a very big variance, which makes the practical application of this
direct method almost infeasible. Thus, variance reduction techniques are in order.
Since in this problem both $h(x)$ and $g(x)$ are fixed, variance reduction cannot be
achieved by choosing appropriate $g(x)$'s. Regenerative techniques are usually used
to reduce the variance at the cost of possibly increasing of bias.

Along the same line is the idea of using a single simulation run to estimate the
derivatives of the system performance measure with respect to a parameter of the
system. Two main approaches in this area are the likelihood ratio method, or the
score function method (3-5), and perturbation analysis (6, 7). In the likelihood ratio
method of estimating the derivative, one uses

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left\{ \frac{\delta g(x_n)}{f(x_n)} \frac{\delta}{\delta \theta} \right\}
$$

(2)

to estimate the derivative of $E[f(y)]$ with respect to a parameter $\theta$. (The subscript $\theta$
is added to indicate the dependency of the density function on $\theta$. The performance
function $f$ is assumed to be independent of $\theta$.)

In this article, we introduce a different approach that applies the idea of impor-
tance sampling to stochastic systems modeled by Markov processes. We call this
approach ensemble-average importance sampling (EAIS). The approach can be ap-
p lied to the derivative estimation, but herein we shall focus on estimating the per-
formance measures of another system with different parameters, as shown in Equa-
tion (1). We shall show that just as in the simple random variable case, one can
obtain the estimates of the steady-state performance of many stochastic systems
(having the same structure but different parameters) with reasonable variances by
observing a single sample path of one of these systems. This method can be used
in parallel computers to simulate the performance of many systems simultaneously.
The main idea of the approach is, instead of estimating the steady-state performance
measures, which are based on the time average of a long sample path, we estimate

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the $K$-step state-transition matrix (the $K$th power of the state-transition matrix) by using the ensemble average. This approach can be viewed as being stimulated by the following observation. There exists a dilemma between the bias of the estimate of the steady-state performance measure and the variance of the estimate of other systems' performance measures. In fact, the longer the simulation run is, the smaller the bias of the estimate of the steady-state performance measure, and the bigger the variances of the estimates of other system performance measures based on the importance sampling method. In the EAIS approach, we do not use the steady-state performance measure, which is based on a long simulation run. Instead, we estimate the $K$-step state-transition matrix, which converges to the steady-state probability as $K$ goes to infinity at a geometrical rate [8]. Thus, we avoid the situation in the standard importance sampling approach, where the steady-state performance measure is obtained by a time average, whereas the performance measures of other systems are obtained by the ensemble average of these time averages. (To distinguish, we shall call the importance sampling applied to a time average the time-average import...

Section 2 discusses the main concept and formulates the problem for Markov chains. Section 3 describes the EAIS algorithms for estimating the steady-state probabilities and performance measures of Markov chains and proves the convergence of the algorithms. The issues regarding the bias and variance are discussed in Section 4. In Section 5, we extend the results to Markov and semi-Markov processes and queuing networks. Simulation examples, two for Markov chains and two for queuing networks, are given to illustrate the biases and the variances of the method. A comparison of the TAIS and EAIS methods is given in Section 6. The article ends with a discussion in Section 7.

2 PROBLEM FORMULATION

Let $X = \{X_0, X_1, \ldots, X_n, \ldots\}$ be a finite-state Markov chain with a state space $\Phi = \{1, 2, \ldots, M\}$. (We distinguish a Markov chain, which is defined on $[0, 1, 2, \ldots]$, from a Markov process, which is defined on $[0, \infty]$.) The state-transition probabilities are $p(i, j), i, j \in \Phi$. Each realization $(x_0, x_1, \ldots, x_n, \ldots)$ is called a sample path of the Markov chain. We assume that $X$ is irreducible and aperiodic, and hence ergodic. Let $(\Omega, \Sigma, \mathcal{P})$ be the underlying probability space of the Markov chain, with $\mathcal{P}$ being the probability measure over the $\sigma$-field $\Sigma$. $\mathcal{P}$ is determined by $p(i, j), i, j \in \Phi$, and the initial probability $P(X_0)$. If the Markov chain always starts with a particular state $i$, then $P(X_0 = i) = 1$ and $P(X_0 = j) = 0$ for all $j \neq i$. Each $\omega \in \Omega$ corresponds to a sample path. Let $\pi(i), i \in \Phi$, be the steady-state probability of the Markov chain (i.e., the invariant measure on the state space).
Let \( f(i), i = 1, 2, \ldots, M, \) be a performance function on the state space \( \Phi. \) The expected value of \( f \) in steady state is \( F = E_x(f) = \sum_{i=1}^{M} f(i) \pi(i), \) where \( E_x \) is the expectation on the state space \( \Phi \) associated with the steady-state probability \( \pi(i). \) By ergodicity, \( F = \lim_{n \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(X_n) \) with probability one (w.p.1). In particular, if we choose \( f \) to be the indicator function \( I(x) = 1 \) for \( X = i \) and \( I(x) = 0 \) for \( X \neq i, \) then
\[
E_x(I(i)) = \pi(i) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} I(X_n) \quad \text{w.p.1. (3)}
\]

In simulation, only sample paths with a finite length can be generated. Let \( N \) be the length (or, precisely, the number of transitions) of the sample paths. We say that all the \( N \)-transition sample paths form an \( N \)-transition Markov chain. Each \( N \)-transition sample path can be viewed as a truncation of some sample paths with an infinite length. Thus, each \( N \)-transition sample path corresponds to a set in \( \Omega, \) and each point in the set corresponds to a sample path having the same first \( N \)-transitions. Let \( \Sigma^N \) be the \( \sigma \)-field generated by the \( N \)-transition Markov chain (i.e., the smallest \( \sigma \)-field with respect to which the sets corresponding to the \( N \)-transition sample paths are measurable). It is clear that \( \Sigma^N \subset \Sigma^{N+1} \) for all \( N > 0. \) To analyze the results of the simulation, we need only the probability space \( (\Omega, \Sigma^N, P), \) which contains all the information about the \( N \)-transition sample paths. For example, if we define
\[
f^N = \frac{1}{N} \sum_{n=1}^{N} f(X_n),
\]
then \( f^N \) is a \( \Sigma^N \) measurable random variable and can be used as an approximation of \( F. \)

Let \( \omega^N \in \Sigma^N \) be the sets corresponding to the \( N \)-transition Markov chain \( \{X_0, X_1, \ldots, X_n\}. \) Then,
\[
P(\omega^N) = P(X_0) \prod_{i=0}^{n-1} p(X_i, X_{i+1}).
\]
This equation defines the probability measure of all the measurable sets in \( \Sigma^N. \) Thus, the expected value of \( f^N \) is
\[
F^N = E(f^N) = \sum_{\omega^N} f^N(\omega^N) P(\omega^N),
\]
where \( E \) is the expectation on \( \Omega \) associated with the probability measure \( P. \) An estimate of \( F^N \) based on \( L \) independent replicas of the \( N \)-transition Markov chain, \( \{X_{0l}, X_{1l}, \ldots, X_{nl}\}, l = 1, 2, \ldots, L, \) is
\[
F^N = \frac{1}{L} \sum_{l=1}^{L} f^N_l,
\]
where \( f^N_l = \frac{1}{N} \sum_{i=1}^{N} f(X_{li}) \) is the performance measured on the \( l \)th replica.

Now we consider another Markov chain \( \Phi' \) with the same state space \( \Phi \) but different transition probabilities \( p'(i, j), i, j \in \Phi. \) The initial probability \( P'(X_0) \) is, however, assumed to be the same as \( P(X_0). \) Without loss of generality, we may also
assume that both X and X' start from the same initial state. The N-transition Markov chain corresponding to this Markov chain X' is defined in the probability space (Ω, Σ, P'). Note that the probability measure P' is induced by p'(i,j), i, j ∈ Φ, and is different from P. The estimate of the expected value of f on the Markov chain X', based on the N-transition Markov chains, is
\[
F^{(N)} = \sum_{\omega} \mathcal{I}^{(N)}(\omega^{(N)}) P'(\omega^{(N)}) = \sum_{\omega} \left[ \mathcal{I}^{(N)}(\omega^{(N)}) \frac{P'(\omega^{(N)})}{P(\omega^{(N)})} \right] P(\omega^{(N)})
\]
\[
= \sum_{\omega} \left\{ \mathcal{I}^{(N)}(\omega^{(N)}) \prod_{k=1}^{N-1} \frac{P'(X_k, X_{k+1})}{P(X_k, X_{k+1})} \right\} P(\omega^{(N)}).
\]
In this equation, we assume that if P(\omega^{(N)}) = 0, then P'(\omega^{(N)}) = 0. Thus, the straightforward application of the importance sampling yields the following estimate of F^{(N)}:
\[
F^{(N)} = \frac{1}{L} \sum_{l=1}^{L} \left\{ \mathcal{I}^{(N)} \prod_{k=1}^{N-1} \frac{P'(X_k, X_{k+1})}{P(X_k, X_{k+1})} \right\},
\]
with \{X_{l0}, X_{l1}, \ldots, X_{lN}\} being the lth replica of the N-transition Markov chain of X. The modifying factor G_N = \prod_{k=1}^{N} \frac{P'(X_k, X_{k+1})}{P(X_k, X_{k+1})} in the term reflects the fact that the probability measure changes from P' to P. The fundamental equation for the T AIS method is Equation (4). In statistics, G_N is called the likelihood ratio.

For a large N, the variance of F^{(N)} may be very big. This can be seen from
\[
\text{var} F^{(N)} = \frac{1}{L} \text{var} \left\{ \mathcal{I}^{(N)} \prod_{k=1}^{N-1} \frac{P'(X_k, X_{k+1})}{P(X_k, X_{k+1})} \right\}.
\]
For a large N, which is usually required for estimating the steady-state performance, the variance of the product on the right-hand side of Equation (5) may be very big (see Section 6 and [3] and [2] for further discussions). On the other hand, L cannot be very large in practice. Therefore, the straightforward important sampling estimate (4), which is based on the Markov chain X, is virtually unusable.

Now we propose another approach that may overcome the above-mentioned difficulty due to a large N. First we consider estimating the steady-state probability π(i). Let Q = [Q(i,j)]_{i,j=1}^{M} be the transition matrix. The conditional probability of X_k = j, given that X_0 = i, is the (i,j)-entry of the K-step transition matrix Q_k, that is,
\[
P(X_k = j | X_0 = i) = \sum_{k-h, h=1}^{K-1} \prod_{l=0}^{h} p(x_l, x_{l+1}), \quad x_0 = i, x_k = j.
\]
Assuming that the Markov chain is irreducible and aperiodic, we have (see, e.g., [9])
\[
\pi(j) = \lim_{K \to \infty} P(X_K = j | X_0 = i).
\]
Let π = (π(1), π(2), \ldots, π(M)) be the vector of steady-state probabilities. Then π is a solution to
\[
\pi = \pi Q.
\]
The convergence of Equation (7) is usually much faster than that of Equation (3). (See Section 4 for a discussion on the convergence rate.) Thus, if we choose

$$\hat{\pi}(j) = P(X_t = j|X_0 = i)$$ (9)

as an estimate of $\pi'(j)$, then a relatively small $K$ can be used (see the examples in Section 3).

Applying importance sampling to Equations (6) and (9) for the Markov chain $X'$, we have

$$\hat{\pi}'(j) = \sum_{x_0, x_1, \ldots, x_t} \left\{ \prod_{i=0}^{t-1} \frac{P'(x_i, x_{i+1})}{p(x_i, x_{i+1})} \prod_{i=0}^{t-1} P(x_i, x_{i+1}) \right\}, \quad i_0 = i, x_0 = j;

= E \left( \prod_{i=0}^{t-1} \frac{P'(x_i, x_{i+1})}{p(x_i, x_{i+1})} \right) .$$

This shows that to obtain $\hat{\pi}'(j)$ on a sample path of $X$ generated according to $p(i, j)$, $i, j = 1, 2, \ldots, M$, a modifying factor $\prod_{i=0}^{t-1} \frac{P'(x_i, x_{i+1})}{p(x_i, x_{i+1})}$ has to be used. An estimate based on $L$ replicas of $X, (X_{i_0}, x_{i_1}, \ldots, x_{i_L}), i = 1, 2, \ldots, L$, is

$$\hat{\pi}'(j) = \frac{1}{L} \sum_{i=1}^{L} \left( \prod_{i=0}^{t-1} \frac{P'(x_i, x_{i+1})}{p(x_i, x_{i+1})} \right), \quad x_{i_0} = i, x_{i_L} = j .$$ (10)

Suppose that $N = 1,000$ is needed to estimate the steady-state probability by using Equation (3), and that $K = 5$ provides a good approximation for $\pi(j)$ in Equation (6). In the straightforward application of the importance sampling, the probability space $(\Omega, \Sigma^{(N)}, P)$ is used. In the approach proposed here, only the space $(\Omega, \Sigma^{(N)}, P)$ is required. The functions measurable on $\Sigma^{(N)}$ are usually "smoother" than those measurable on $\Sigma^{(M)}$.

The main distinction between Equations (3) and (6) is as follows. Equation (3) is a time average, which usually requires many transitions and results in a big variance. Conversely, Equation (6) represents an ensemble average, which uses the sum $\sum_{x_0, x_1, \ldots, x_t}$ over the probability space. More specifically, the TAIS estimate [Equation (4)] changes the probability measure for each $N$-transition sample path, which is a fundamental set in $\Omega^{(N)}$, whereas the EAIS estimate [Equation (10)] changes the probability measure for each event representing the system transition from $X_0 = i$ to $X_0 = j$, which is a set in $\Omega^{(N)}$. Loosely speaking, the TAIS method updates the estimate after obtaining the average number of the event $X_0 = j$ in an $N$-transition sample path, whereas the EAIS method updates the estimate whenever the single event $X_0 = j$ occurs. Thus, the ensemble-average form is more suitable for the importance sampling method.

3 ENSEMBLE-AVERAGE IMPORTANCE SAMPLING ALGORITHMS AND THEIR CONVERGENCE

Although Equation (10) represents the ensemble average, it still can be estimated based on a single sample path of the Markov chain $X$. Suppose $X_0 = i$, and let

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\[ n_i = \sum_{n=0}^{N-K} I(X_n) \]

and

\[ n_{ij} = \sum_{n=0}^{N-K} I(X_n) I(Y(X_n, z)). \]

Then \( \hat{\pi}(i, j) = n_{ij}/n_i \) can be used as an estimate of \( P(X_j = j|X_i = i) \). Furthermore, from Equation (7), \( P(X_j = j|X_i = i) \) converges to \( \pi(j) \), which does not depend on \( i \). Thus, as \( K \) increases, \( \hat{\pi}(i, j) \) becomes almost independent of \( i \), or equivalently, \( n_{ij}/n_i \)'s become almost the same for different \( i \)'s. Thus, we can use the following estimate:

\[ \hat{\pi}(j) = \frac{\sum_{n=0}^{N-K} n_{ij}}{\sum_{n=0}^{N-K} n_i} = \frac{\sum_{n=0}^{N-K} n_{ij}}{N - K + 1} \]

\[ = \frac{1}{N - K + 1} \sum_{n=0}^{N-K} \sum_{a=0}^{N-K} I(X_n) I(Y(X_n, z)) \]

\[ = \frac{1}{N - K + 1} \sum_{n=0}^{N-K} I(X_n) I(Y(X_n, z)) \]

This is the same as Equation (3). The way we apply importance sampling is, however, different from that of TADS.

In EAIS, we do not apply importance sampling directly to \( \hat{\pi}(j) \). Instead, we apply importance sampling to \( n_{ij} \). First, we have the following estimate for \( I_i(X_n) I(Y(X_n, z)) \) on the Markov chain \( X' \):

\[ I_i(X_n) I(Y(X_n, z)) = \prod_{k=1}^{K-1} \frac{p'(X_{n+k}, X_{n+k+1})}{p(X_{n+k}, X_{n+k+1})} \]

where \( X_n \) and \( X_{n+k} \) are measured on \( X \). Summing up over \( n \), we get the estimate for \( n_{ij} \). Finally, the estimate for \( \pi'(j) \) is

\[ \hat{\pi}'_K(j) = \frac{\sum_{n=0}^{N-K} n_{ij}}{N - K + 1} \]

\[ = \frac{1}{N - K + 1} \sum_{n=0}^{N-K} \sum_{a=0}^{N-K} I(X_n) I(Y(X_n, z)) \]

\[ \frac{p'(X_{n+k}, X_{n+k+1})}{p(X_{n+k}, X_{n+k+1})} \] (11)

To implement Equation (11) in simulation, one needs a register of length \( K \) to store the value of \( X_{n+k}, k = 0, 1, \ldots, K - 1 \). However, the calculation of the product on the right-hand side of Equation (11) can be simplified. Let \( r_{n+k} = p'(X_{n+k}, X_{n+k+1})/p(X_{n+k}, X_{n+k+1}) \) and \( w_{n+k} = \prod_{k=0}^{K-1} r_{n+k+1} \). Then

\[ w_{n+k+1} = w_{n+k} \frac{r_{n+k+1}}{r_{n+k}} \] (12)
Thus, only one multiplication and one division, instead of \( K \) multiplications, are needed to calculate the modifying factor at each transition.

The EAM algorithm based on Equation (11) simply applies the following steps on a sample path of a Markov chain.

**Algorithm 1 (for steady-state probabilities of Markov chains)**

1. At the initial state, set \( d(i) := 0, i = 1, 2, \ldots, M, \) and \( w_0 := 1. \)
2. At the \( n \)th transition,
   (a) if \( n = K, \) set \( w_n := w_{n-1}r_{i-1}, \)
   (b) if \( n > K, \) set \( w_n := w_{n-1}r_{i-1}/r_{i-K-1} \) and \( d(X_n) := d(X_n) + w_n. \)
3. At the end of the simulation (i.e., the \( N \)th transition), set \( \bar{d}_K(i) = \frac{d(i)}{N - K + 1}, i = 1, 2, \ldots, M. \)

**Remark 1.** In this algorithm, the values for state \( i \) are collected in \( d(i) \) whenever \( X_n = i \) holds. In this way, two or more consecutive values may be correlated. For example, if \( X_n = X_m = i \) and \( 0 < n_1 - n < K, \) then \( w_{n_1} \), based on \( X_{n_1-i}, \ldots, X_m, \) and \( w_{n_1} \), based on \( X_{n_1-i}, \ldots, X_m, \) contain some common states, \( X_{n_1-i}, \ldots, X_m. \) This correlation may result in an increasing of the variance of the estimate. One possible modification is to collect \( d(i) \) only at points that are \( K \) transitions apart. This, however, increases the length of the simulation for collecting the same number of points.

**Remark 2.** For periodic Markov chains, \( Q^T \) does not converge. Therefore, Equation (10) cannot be used as an estimate of the steady-state probability. In addition, if for some reason (e.g., due to the periodicity of the random number generator in a simulation) a sample path possesses some periodicity, then on the sample path, \( X_1, \ldots, X_{n+K}, \) may not go through all the possible paths, and the variance may increase. This is similar to the systematic sampling technique in simulation (see, e.g., [10] and [11]) where periodicity increases the variance.

The convergence of the algorithm is established in the following theorem.

**Theorem 1**

Consider the two irreducible and aperiodic Markov chains \( X \) and \( X' \) having the same finite state space \( \{1, 2, \ldots, M\}, \) but different transition probabilities \( p(i, j) \) and \( p'(i, j), \) \( i, j = 1, 2, \ldots, M, \) respectively. Assume that if \( p(i, j) = 0 \) then \( p'(i, j) = 0. \) Let \( (X_0, X_1, \ldots, X_N) \) be a sample path of \( X \) and \( \bar{d}_K(i) \) be as defined in Equation (11). Then we have

\[
\lim_{K \to \infty} \bar{d}_K(i) = \sum_{i=1}^{M} \pi(i) P'(X_{n+1} = j | X_n = i) \quad \text{w.p.1}\]

(13)

and

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\[ \lim_{K \to \infty} \lim_{N \to \infty} \Phi_{k,K}(j) = \pi'(j). \]  

(14)

Proof. We first assume \( p(i,j) > 0 \) for all \( i, j \). Thus all the fractions in \( \Phi_{k,K} \) are well defined. Let

\[ \rho_{k,K}(i,j) = \frac{1}{N - K + 1} \sum_{X_0} \left[ I_j(X_0)I_j(X_{(K+1)}) \prod_{s=0}^{K-1} \frac{P'(X_{s+1}, X_{s+2})}{P(X_{s+1}, X_{s+2})} \right]. \]

Then

\[ \Phi_{k,K}(j) = \sum_{i=1}^{N} \rho_{k,K}(i,j). \]

The irreducible and aperiodic Markov chain \( X \) is ergodic [9]. Let

\[ Y_n(X_0, X_1, \ldots, X_{n+2}) = \frac{I_j(X_0)I_j(X_2)}{P(X_1, X_2)}, \]

then \( Y(X_0, X_1, \ldots, X_{n+2}) \) is a measurable function of \( X_0, \ldots, X_{n+2} \). Thus, \( \{Y_0, Y_1, \ldots\} \) is also ergodic [12]. We have

\[ \lim_{N \to \infty} \rho_{k,K}(i,j) = \lim_{N \to \infty} \frac{1}{N - K + 1} \sum_{X_0} Y(X_0, \ldots, X_{n+2}) = \mathbb{E}[Y(X_0, \ldots, X_{n+2})] \quad \text{w.p.1} \]

Furthermore,

\[ \mathbb{E}[Y(X_0, \ldots, X_{n+2})] \]

\[ = E \left[ I_j(X_0)I_j(X_2) \prod_{s=0}^{K-1} \frac{P'(X_{s+1}, X_{s+2})}{P(X_{s+1}, X_{s+2})} \right] \]

\[ = \pi(i)E \left[ \left( \prod_{s=0}^{K-1} \frac{P'(X_{s+1}, X_{s+2})}{P(X_{s+1}, X_{s+2})} \right) \mid X_0 = i \right] \]

\[ = \pi(i) \sum_{x_1, x_2, \ldots, x_{n+2}} \left( \prod_{s=0}^{K-1} \frac{P'(X_{s+1}, X_{s+2})}{P(X_{s+1}, X_{s+2})} \prod_{s=0}^{K-1} \frac{P(x_s, x_{s+1})}{P(x_s, x_{s+1})} \right) ; \quad x_0 = i, x_{n+2} = j \]

\[ = \pi(i) \sum_{x_1, x_2, \ldots, x_{n+2}} \left( \prod_{s=0}^{K-1} P'(X_{s+1}, X_{s+2}) \right) ; \quad x_0 = i, x_{n+2} = j \]

\[ = \pi(i)P'(X_{n+2} = j \mid X_0 = i). \]

Equation (13) follows directly by summing up the above equation. Using (13) and \( \lim_{K \to \infty} P'(X_{n+2} = j \mid X_0 = i) = \pi'(j) \), we obtain Equation (14).

Now assume that \( p(i,j) = 0 \) for some \( i \) and \( j \). In this case, any sample path of \( X \) will not contain transitions from state \( i \) to state \( j \). \( \Phi_{k,K}(j) \) is still well defined on all the sample paths of \( X \). Furthermore, if \( p'(i,j) = 0 \), then

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\[
\sum_{x_{n-1}, x_n} \left( \prod_{k=0}^{K-1} p'(x_{k+1}, x_{k+2}) \right) \]  

(15)

obtained from sample paths of \( X \) via

\[
\sum_{x_{n-1}, x_n} \left( \prod_{k=0}^{K-1} p'(x_{k+1}, x_{k+2}) \right) \prod_{k=0}^{K-1} p(x_{k+1}, x_{k+2})
\]

represents all the possible sample paths of \( X' \) on which \( X_n = i \) and \( X_{n+1} = j \). Therefore, with \( X_{n-1} = i \) and \( X_n = X_{n+1} = j \), it holds that, in this case,

\[
\pi(i) = \sum_{x_{n-1}, x_n} \left( \prod_{k=0}^{K-1} p'(x_{k+1}, x_{k+2}) \right) = \pi(i)p'(X_{n+1} = j/X_n = i).
\]

(16)

However, if \( p'(i_1, j_1) \neq 0 \), then the sample path of \( X' \) containing the transition from \( i \) to \( j \) is not included in expression (15), which is a sum over all possible sample paths of \( X \). Thus, Equation (16) may not hold; that is, Equations (15) and (14) may not hold. This completes the proof.

Finally, for any performance function \( f(i), i = 1, 2, \ldots, M \), we can estimate the steady-state performance \( F = E'_\pi(f) = \Sigma_M f(i)\pi'(i) \), where \( E'_\pi \) is the expectation associated with \( \pi'(i) \) on \( \Phi \), by \( \hat{F}_{n,K} \) specified in the following algorithm, which is a simple modification of Algorithm 1.

**Algorithm 2 (for steady-state performance measures of Markov chains)**

1. At the initial state, set \( d := 0 \) and \( w_0 := 1 \).
2. At the \( n \)th transition,
   (a) if \( n = K \), set \( w_n := w_{n-1}/p_n \);  
   (b) if \( n > K \), set \( w_n := w_{n-1}/p_n r_{n-1,X_{n-1}=X} \) and \( d := d + f(X_n)w_n \).
3. At the end of the simulation (i.e., the \( N \)th transition), set \( \hat{F}_{n,K} = \frac{d}{N - K + 1} \).

Before ending this section, we provide two simulation examples.

**Example 1**

In this example, we study a three-state Markov chain. The transition matrices and the simulation results are shown in Tables 1–3. In the simulation, we choose \( K = 3 \) and \( N = 100,000 \). \( \pi(i) \) and \( \pi'(i) \) are the steady-state probabilities for the Markov chains \( X \) and \( X' \). \( \pi'(i) \) is the direct simulation estimate for the steady-state probability of \( X \) based on a sample path of \( X \). \( \pi'(i) \) is the EAIS estimate of the steady-state probability of the Markov chain \( X' \) based on a sample path of \( X \). Note that in the example \( p'(1, 1) = p'(2, 2) = p'(3, 1) = 0 \), whereas the corresponding transition probabilities for \( X \) are nonzero. This shows that although Algorithm 1 does not apply to cases where \( p(i, j) = 0 \) and \( p'(i, j) \neq 0 \) for some \( i \) and \( j \), it does apply to cases where \( p(i, j) = 0 \) but \( p'(i, j) = 0 \).
Table 1. State-Transition Matrix $Q = \{p(i, j)\}$ for Example 1

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.1</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.6</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 2. State-Transition Matrix $Q' = \{p'(i, j)\}$ for Example 1

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.0</td>
<td>0.6</td>
</tr>
<tr>
<td>3</td>
<td>0.0</td>
<td>0.8</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 3. Simulation Results of Example 1

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theor. $\pi(i)$</td>
<td>0.340</td>
<td>0.355</td>
<td>0.305</td>
</tr>
<tr>
<td>Simul. $\hat{\pi}(i)$</td>
<td>0.339</td>
<td>0.355</td>
<td>0.306</td>
</tr>
<tr>
<td>Theor. $\pi'(i)$</td>
<td>0.170</td>
<td>0.426</td>
<td>0.404</td>
</tr>
<tr>
<td>EAIS $\hat{\pi}'(i)$</td>
<td>0.165</td>
<td>0.434</td>
<td>0.401</td>
</tr>
</tbody>
</table>

Example 2

A five-state Markov chain is studied in this example. The notations are the same as those in Example 1, and we listed the results for $K = 3$ and $N = 100,000$. Other parameters and simulation results are shown in Tables 4—6.

4 BIAS AND VARIANCE REDUCTION

Because the convergence of Equation (7) is much faster than that of the time average in Equation (3), $K$ in Equation (7) can be chosen much smaller than $N$ in Equation...
### Table 4. State-Transition Matrix $Q = [p(i, j)]$
for Example 2

<table>
<thead>
<tr>
<th>$(i, j)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20</td>
<td>0.15</td>
<td>0.15</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>0.30</td>
<td>0.20</td>
<td>0.20</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>3</td>
<td>0.20</td>
<td>0.05</td>
<td>0.05</td>
<td>0.30</td>
<td>0.40</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
<td>0.20</td>
<td>0.20</td>
<td>0.15</td>
<td>0.35</td>
</tr>
<tr>
<td>5</td>
<td>0.40</td>
<td>0.10</td>
<td>0.00</td>
<td>0.35</td>
<td>0.15</td>
</tr>
</tbody>
</table>

### Table 5. State-Transition Matrix $Q' = [p'(i, j)]$
for Example 2

<table>
<thead>
<tr>
<th>$(i, j)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20</td>
<td>0.25</td>
<td>0.10</td>
<td>0.10</td>
<td>0.35</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>0.50</td>
<td>0.10</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>3</td>
<td>0.50</td>
<td>0.05</td>
<td>0.05</td>
<td>0.10</td>
<td>0.30</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
<td>0.25</td>
<td>0.20</td>
<td>0.30</td>
<td>0.10</td>
</tr>
<tr>
<td>5</td>
<td>0.20</td>
<td>0.10</td>
<td>0.00</td>
<td>0.50</td>
<td>0.20</td>
</tr>
</tbody>
</table>

### Table 6. Simulation Results of Example 2

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theor. $\pi(i)$</td>
<td>0.241</td>
<td>0.145</td>
<td>0.120</td>
<td>0.242</td>
<td>0.252</td>
</tr>
<tr>
<td>Simul. $\hat{\pi}(i)$</td>
<td>0.241</td>
<td>0.144</td>
<td>0.120</td>
<td>0.243</td>
<td>0.252</td>
</tr>
<tr>
<td>Theor. $\pi'(i)$</td>
<td>0.183</td>
<td>0.264</td>
<td>0.111</td>
<td>0.242</td>
<td>0.201</td>
</tr>
<tr>
<td>EAIS $\hat{\pi}'(i)$</td>
<td>0.183</td>
<td>0.261</td>
<td>0.111</td>
<td>0.245</td>
<td>0.201</td>
</tr>
</tbody>
</table>

(3). In fact, to obtain the steady-state probability $\pi(i)$, at least $N > 1,000$ (say, for $M = 3$) is needed. Such a large $N$ leads to a big variance, making the TAIS method infeasible in practice. Conversely, the numerical examples in Section 3 indicate that

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in the EAIIS algorithm $K = 2$ leads to very good results for Markov chains with five states (i.e., $M = 5$).

It is known that Equation (7) converges at a geometrical rate, that is, there exist an $A > 0$ and a $\rho$, $0 < \rho < 1$, such that (see, e.g., the appendix of [9])

$$|\pi(j) - P(X_k = j|X_0 = i)| \leq A \rho^k,$$

where $\rho$ is the largest absolute value of the eigenvalues of the transition matrix $Q$ excluding the eigenvalue 1, and $A$ can be determined by the eigenvectors of $Q$. This equation gives an upper bound of the bias due to a finite $K$. In Example 1, the three eigenvalues of the transition matrix are $1$, $-0.373$, and $0.027$. Thus, $\rho = 0.373$. For $K = 3$, we have $\rho^3 = 0.05$. This implies that if $K' = K + 3$, then the bias corresponding to $K'$ is only 5% of that corresponding to $K$. The bias decreases rapidly as $K$ increases.

There is, however, a trade-off between the variance and bias. The estimate $\hat{\pi}(i)$ with a finite $K$, just as the TAIS estimate with a finite $N$, is biased. The larger $K$ is, the smaller the bias, and the bigger the variance. With a small bias and a big variance, it requires a long simulation run to obtain an accurate estimate, which is close to the real value. Conversely, with a big bias and a small variance, it requires only a short run to obtain an estimate of the mean, which may be apart from the real value. Thus, if the length of a simulation is fixed, both variance and bias will affect the accuracy of the estimate from one simulation. Therefore, $K$ has to be properly chosen before running the simulation. Table 7 gives a comparison of the

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(i)$</td>
<td>0.183</td>
<td>0.264</td>
<td>0.111</td>
<td>0.242</td>
<td>0.201</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 2$</td>
<td>0.183</td>
<td>0.251</td>
<td>0.112</td>
<td>0.249</td>
<td>0.204</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 3$</td>
<td>0.183</td>
<td>0.261</td>
<td>0.111</td>
<td>0.245</td>
<td>0.201</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 4$</td>
<td>0.181</td>
<td>0.265</td>
<td>0.111</td>
<td>0.243</td>
<td>0.200</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 5$</td>
<td>0.182</td>
<td>0.269</td>
<td>0.111</td>
<td>0.241</td>
<td>0.198</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 10$</td>
<td>0.187</td>
<td>0.261</td>
<td>0.113</td>
<td>0.235</td>
<td>0.204</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 15$</td>
<td>0.187</td>
<td>0.267</td>
<td>0.121</td>
<td>0.210</td>
<td>0.214</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 20$</td>
<td>0.213</td>
<td>0.258</td>
<td>0.121</td>
<td>0.207</td>
<td>0.201</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 30$</td>
<td>0.207</td>
<td>0.272</td>
<td>0.103</td>
<td>0.215</td>
<td>0.202</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 50$</td>
<td>0.219</td>
<td>0.270</td>
<td>0.078</td>
<td>0.286</td>
<td>0.147</td>
</tr>
<tr>
<td>$\hat{\pi}(i), K = 100$</td>
<td>0.273</td>
<td>0.151</td>
<td>0.067</td>
<td>0.286</td>
<td>0.243</td>
</tr>
</tbody>
</table>
results for the Markov chain discussed in Example 2 with different values of \( K \). The table shows that, even for \( K = 2 \), the results are very accurate, and that as \( K \) increases to 20, the results start to deteriorate. This indicates that the accuracy is not very sensitive to the value of \( K \) and that the variance is a more crucial issue than the bias. The length of the simulations in Table 7 are fixed at \( N = 100,000 \). When \( K > 20 \), it is the big variance that makes \( N = 100,000 \) insufficient for an accurate estimate. (As one referee pointed out, the bigger \( K \) is, the stronger the correlation between consecutive values collected in Algorithm 1; this strong correlation may also contribute to a big variance for \( K > 20 \).)

The variance of the estimate can be used as a measure of how close the transition matrix \( \mathcal{Q}' \) is to the transition matrix \( \mathcal{Q} \). It is important to develop a practical method providing us the variance for a given \( K \). However, this is a difficult task, and, so far, we do not have analytical formulas for the variance of the estimate shown in Equation (10). Of course, the values of the estimate can be obtained numerically. Section 6 and [3] and [2] contain further discussions about the variance of the likelihood ratio estimates.

In the following, we propose an iterative method that may reduce the bias. As shown in Equation (13), \( \hat{\pi}_{k,x}^i(j) \) is in fact an estimate of \( \Sigma_{n+1} \pi(i)P(X_{n+1} = j|X_n = i) \) instead of that of \( P'(X_{n+1} = j|X_n = i) \). Let

\[
\hat{\pi}_{k,x}^i(j) = \frac{\sum_{n=0}^{\infty} \prod_{r=0}^{k-1} P'(X_r, X_{r+1}) \tilde{f}_r(X_r, X_{r+1})}{\sum_{n=0}^{\infty} \tilde{f}_r(X_r, X_{r+1})}.
\]

Similar to Equation (13), we can prove

\[
\lim_{n \to \infty} \pi_{n,x}^i(j) = P'(X_{n+1} = j|X_n = i) \quad \text{w.p.1.} \tag{17}
\]

Based on Equation (17), an EAIS algorithm can be developed. Both \( \hat{\pi}_{k,x}^i(j) \) and \( \hat{\pi}_{k,x}^i(j) \) provide biased estimates for \( \pi'(j) \). The latter is a weighted average of the former. Since one sample path provides all the \( \hat{\pi}_{k,x}^i(j) \)'s for \( i, j = 1, 2, \ldots, M \), it is more efficient to use all these estimates in a weighted average form. Now, what is the best weighting factor? To answer this question, we let

\[
\hat{\pi}_{k,x}^i(j) = \sum_{i=1}^{M} \alpha_i \hat{\pi}_{k,x}^i(j),
\]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M) \) and \( \alpha_i \geq 0, i = 1, 2, \ldots, M \), are weighting factors satisfying \( \Sigma_{i=1}^{M} \alpha_i = 1 \). As \( N \to \infty \), \( \hat{\pi}_{k,x}^i(j) \) is a strongly consistent estimate of \( \Sigma_{i=1}^{M} \alpha_i P'(X_{n+1} = j|X_n = i) \), that is,

\[
\lim_{n \to \infty} \hat{\pi}_{k,x}^i(j) = \sum_{i=1}^{M} \alpha_i P'(X_{n+1} = j|X_n = i) \quad \text{w.p.1.}
\]

In \( \pi_{k,x}^i(j) \), we have \( \alpha_i = \pi(i) \). Let

\[
b_{ij} = P'(X_{n+1} = j|X_n = i) - \pi'(j)
\]

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be the bias of \( \hat{\theta}_{i,k}(j|i) \). Then the bias of \( \hat{\theta}_{i,k}^*(j) \) is

\[
\sum_{i=1}^{N} \alpha_i P'(X_{i+k} = j|X_0 = i) - \pi'(j)
\]

\[
= \sum_{i=1}^{N} [\alpha_i - \pi'(i)] P'(X_{i+k} = j|X_0 = i) - \pi'(j)]
\]

\[
= \sum_{i=1}^{N} [\alpha_i - \pi'(i)] B_{ij}.
\]

(18)

From Equation (18), the bias of \( \hat{\theta}_{i,k}^*(j) \) would be zero if we choose \( \alpha_i = \pi'(i) \). Of course, this is impossible because \( \pi'(j) \) is just what we want to estimate. In \( \hat{\theta}_{i,k}(j) \), we choose \( \alpha_i = \pi(i) \). Thus, if \( \pi'(i)'s \) are close to \( \pi(i)'s \), \( \hat{\theta}_{i,k}(j) \) is a better estimate than \( \hat{\theta}_{i,k}(j, t) \) in the sense that it may have a smaller bias.

Next, for a fixed \( j \), the biases \( (b_i)'s \) for different \( t \)'s may contain a common term, defined as \( e_j \). More precisely, let \( b_i = \delta_{ij} + e_j \), where \( e_j \) can be viewed as the common bias contained in all \( (b_i)'s \), and the \( (b_i)'s \) measure the difference among \( (b_i)'s \) for different \( t \)'s. From Equation (18), the bias of \( \hat{\theta}_{i,k}^*(j) \) is (note that \( \sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{M} \pi'(i) = 1 \))

\[
\sum_{i=1}^{N} [\alpha_i - \pi'(i)] B_{ij} + e_j \approx \sum_{i=1}^{N} [\alpha_i - \pi'(i)] B_{ji}.
\]

This shows that the weighted average \( \hat{\theta}_{i,k}^*(j) \) or \( \hat{\theta}_{i,k}(j) \) 'averages out' the common bias contained in \( \hat{\theta}_{i,k}(j|i) \).

From the preceding discussion, if we have some a priori knowledge about \( \pi'(j) \), we can reduce the bias by choosing the weighting factors \( \alpha_j \) to be the a priori estimate of \( \pi'(j) \). Furthermore, after simulation the estimates of \( \pi'(j)'s \) are improved and can be used as a new set of weighting factors. This suggests that an iterative procedure can be developed to reduce the bias. Let \( \alpha_0 = (\alpha_{01}, \ldots, \alpha_{0M}) \) be the a priori estimate of \( \pi'(j) \), which may be \( \pi(j) \) or any other guessed values. We use the \( (l - 1) \)th estimate of \( \pi'(j) \), \( \hat{\pi}_{i,k}^{(l-1)}(j) \), as the weighting factor for the \( l \)th estimate. That is, we set

\[
\alpha_{ij} = \hat{\pi}_{i,k}^{(l-1)}(j) := \sum_{i=1}^{M} \alpha_{i-1,j} \hat{\theta}_{i,k}(j|i).
\]

This bias reduction procedure is, in fact, a combination of the simulation and analytical computation. For \( K = 1 \), this procedure reduces to the analytical method of calculating \( Q' = Q_{i+1} \times Q \), \( Q = [p'(i,j)]_{i,j} \), where \( M \times M \) variables \( \hat{\theta}_{i,k}(j|i) \), \( i, j = 1, 2, \ldots, M \), and hence is not very practical. Note that in estimating the performance measure \( E_{\alpha}(f) \) in simulation using EAIS, even the \( M \) variables \( \hat{\theta}_{i,k}(j) \) need not be stored. Although online updating algorithms based on this iterative procedure may be developed, we will not discuss this procedure further.
5 EXTENSIONS TO MARKOV AND SEMI-MARKOV PROCESSES
AND QUEUEING NETWORKS

The extension of the results in the previous sections to Markov processes is almost straightforward. Consider a Markov process $Y = \{Y_t; t \in [0, \infty)\}$ with the state space $\Phi = \{1, 2, \ldots, M\}$ and an ergodic embedded Markov chain $X = \{X_0, X_1, \ldots, X_s, \ldots\}$. Let $T_n$ be the $n$th transition time of the Markov process and $S_n = T_{n+1} - T_n$. We assume that $Y$ is right-continuous. Then we have $X_n = Y_{T_n}$. Let $p(i, j), i, j = 1, 2, \ldots, M$ be the transition matrix of $X$. Note that $p(i, i) = 0$ for all $i$, since $X$ is an embedded Markov chain. Let $\lambda_n, 0 < \lambda_n < \infty, i = 1, 2, \ldots, M$, be the transition rate at state $i$; and let $\mu_n = 1/\lambda_n = E(S_n | X_n = i), i = 1, 2, \ldots, M$, be the mean sojourn time at state $i$. Then [9]

$$P(X_{n+1} = j, T_{n+1} - T_n \leq u | X_n = i) = p(i, j)(1 - e^{-\lambda_n u}).$$

Let $(\pi(1), \pi(2), \ldots, \pi(M))$ be the steady-state probability of the embedded chain $X$ [i.e., the solution to Equation (8) with $\sum_{i=1}^{M} \pi(i) = 1$]. Then the steady-state probability of the Markov process $Y$ is [9]

$$p(i) = \lim_{n \to \infty} P(Y_n = i) = \frac{\sum_{i=1}^{M} \pi(i)\mu_i}{\sum_{i=1}^{M} \pi(i)\mu_i}. \quad (19)$$

Now consider another Markov process $Y'$ with the same state space, the same mean sojourn time $\mu_1, 1, 2, \ldots, M$, and different transition probabilities $p'(i, j), i, j = 1, 2, \ldots, M$. The steady-state probability of the Markov chain $X'$ embedded in $Y'$, $\pi'(i), i = 1, 2, \ldots, M$, can be estimated by using the EABS algorithm described in Section 2. Thus, using Equation (19) we can develop the following algorithm for estimating the steady-state probability of the Markov process $Y', p'(i), i = 1, 2, \ldots, M$, based on a sample path of $Y$.

Algorithm 3 (for steady-state probabilities of Markov processes)

1. At the initial state, set $d(i) := 0, i = 1, 2, \ldots, M$, and $w_0 := 1$.
2. At the $n$th transition,
   (a) if $n \leq K$, set $w_n := w_{n-1}r_{n-1}\lambda_n$;
   (b) if $n > K$, set $w_n := w_{n-1}r_{n-1}\lambda_n - r_{n-1}\mu_n$ and $d(X_n) := d(X_n) + S_nw_n$.
3. At the end of the simulation (i.e., the $N$th transition) set $\beta(i) = \frac{d(i)}{\sum_{i=1}^{M} d(j)}$, $i = 1, 2, \ldots, M$.

In the algorithm, $w_n, r_n$, and $K$ are defined in Section 3, and $\beta(i)$ is the estimate of $p'(i)$ based on an $N$-transition sample path of $Y$. From Theorem 1, the convergence of this algorithm is obvious.

Now, suppose we are interested in estimating the steady-state expected value of a performance measure

$$F = \sum_{i=1}^{M} f(i)p'(i) = \lim_{N \to \infty} \frac{1}{T_f} \int_0^{T_f} f(Y_t) \, dt \quad \text{w.p.1.}$$

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We can use the following algorithm to obtain the estimate \( \hat{F}_{lx} \), which is based on an \( N \)-transition sample path of the Markov process \( Y \).

**Algorithm 4 (for steady-state performance measures of Markov processes)**

1. At the initial state, set \( d := 0 \), \( c := 0 \), and \( w_x := 1 \).
2. At the \( n \)th transition,
   (a) if \( n \leq K \), set \( w_x := w_{x-1} r_{x-1} \).
   (b) if \( n > K \), set \( w_x := w_{x-1} r_{x-1} \). Also, set \( d := d + S_x w_x \) and \( c := c + f(X_x) X_x w_x \).
3. At the end of the simulation (i.e., the \( N \)th transition), set \( \hat{F}_{lx} = c/d \), \( i = 1, 2, \ldots, M \).

Algorithms 3 and 4 can also be applied to semi-Markov processes with \( s_i \) being the mean sojourn time at state \( i \), \( i = 1, 2, \ldots, M \). Recall that for a semi-Markov process \( Y \), we have a semi-Markov kernel [9]

\[
R(i, j, t) = P(X_{t+1} = j, T_{t+1} = T_i \leq t | X_t = i), \quad i, j = 1, 2, \ldots, M, t \in [0, \infty).
\]

The semi-Markov kernel can be decomposed into

\[
R(i, j, t) = p(i, j) G(i, j, t)
\]

with \( p(i, j) \) being the transition probability of the embedded Markov chain \( X \) and

\[
G(i, j, t) = P(T_{t+1} = T_i \leq t | X_t = i, X_{t+1} = j), \quad i, j = 1, 2, \ldots, M, t \in [0, \infty).
\]

If \( G(i, j, t) = 1 - e^{-\lambda t} \), then \( Y \) reduces to a Markov process. Let

\[
H(i, j, t) = 1 - \sum_{j' \neq j} G(i, j', t).
\]

Then \( s_i = \int_0^\infty H(i, j, t) dt \) is the mean sojourn time at state \( i \). From Theorem 5.22 of [9], the steady-state probability is also expressed in Equation (19). The EIS Algorithms 3 and 4, therefore, also provide estimates of the steady-state probabilities for semi-Markov processes.

Next, let us consider queueing networks. The state process of a queueing network with exponential service distributions (Jackson type) is a Markov process. Thus, Algorithms 3 and 4 can be applied directly to such networks. Some special features, however, deserve attention. In a closed Jackson network consisting of \( M \) servers, let \( \mathbf{n} = (n_1, n_2, \ldots, n_M) \) be the system state, where \( n_i, i = 1, 2, \ldots, M \), is the number of customers at server \( i \). (We slightly abuse the notation by using \( M \), which has been used for the number of states, to denote the number of servers here.) The routing probabilities of the network are denoted by \( q(i, j) \), \( i, j = 1, 2, \ldots, M \). Let \( \mu_i, i = 1, 2, \ldots, M \), be the mean service rate of server \( i \). Although the number of states of the network is much bigger than \( M \), the state-transition probabilities can be determined by all these \( \mu \)'s and \( q(i, j) \)'s. For example, let \( n' = (\ldots, n_j + 1, \ldots, n_i - 1, \ldots), n_i > 0 \), be a neighboring state of \( n \). Then the transition rate from \( n \) to \( n' \) is

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\[
p(n, n') = \frac{\mu_i}{\sum_{j \in E(n)} \mu_j} q(i, j),
\]
where \(E(n) = 1\) if \(n > 0\) and \(E(n) = 0\) if \(n = 0\). Now, suppose that the routing probabilities change to \(q'(i, j), i, j = 1, 2, \ldots, M\), and that the mean service times remain unchanged. Then

\[
\frac{p'(n, n')}{p(n, n')} = \frac{q'(i, j)}{q(i, j)}.
\]

(20)

This relation shows that to estimate the performance with different routing probabilities, one has to keep only a record of customer transitions instead of state transitions. This simplifies the algorithm.

Suppose at the \(n\)th transition, a customer moves from server \(u_k\) to server \(u_k\). To utilize Equation (20) in the algorithm, we use a register of length \(K\) to store both \(u_{k+1}\) and \(u_{k+1}\), \(k = 0, 1, \ldots, K - 1\) (instead of the numbers of customers at all servers). Let \(r_{k+1} = q'(u_k, u_k)[q(u_k, u_k)]\). With this modification, \(w_k\)'s are defined in the same way as in Equation (12). Let \(f(n)\) be the performance function, \(p(n)\) the steady-state probability, and \(F = \sum_{n=1}^{\infty} f(n)p(n)\) the performance measure. Finally, let \(X_n\) be the \(n\)th state of the embedded Markov chain. Then the EADS algorithm for estimating \(F\) with different routing probabilities is the same as Algorithm 4, with the above modifications of notations. One can see that the algorithm is very efficient when applied to estimate the performance of many networks. Only one sample path has to be simulated.

We present the following two examples to illustrate the accuracy of the results.

Example 3

Consider two three-server closed Jackson networks. Each system consists of five customers. The mean service times are the same for both systems: \(s_1 = 5\), \(s_2 = 10\), and \(s_3 = 7\). The routing probabilities of system A are \(p(1, 1) = p(2, 2) = p(3, 3) = 0.0867\), \(p(1, 2) = p(2, 3) = p(3, 1) = 0.0988\), \(p(2, 1) = p(3, 2) = p(1, 3) = 0.0917\)

<table>
<thead>
<tr>
<th>Sys</th>
<th>Server</th>
<th>Simulated TP</th>
<th>Estimated TP</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td>1</td>
<td>0.0867 (0.0004)</td>
<td>0.0877 (0.0023)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0917 (0.0005)</td>
<td>0.0917 (0.0007)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0988 (0.0006)</td>
<td>0.1004 (0.0009)</td>
</tr>
<tr>
<td>B.</td>
<td>1</td>
<td>0.0942 (0.0005)</td>
<td>0.0948 (0.0025)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0839 (0.0003)</td>
<td>0.0841 (0.0009)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.1152 (0.0005)</td>
<td>0.1150 (0.0008)</td>
</tr>
</tbody>
</table>

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Table 9. Queue-Length (QL) Estimation with $K = 50$

<table>
<thead>
<tr>
<th>Sys</th>
<th>Svr</th>
<th>Simulated QL</th>
<th>Estimated QL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0.831 (0.007)</td>
<td>0.832 (0.031)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.368 (0.021)</td>
<td>3.352 (0.060)</td>
</tr>
<tr>
<td>A.</td>
<td></td>
<td>1.803 (0.017)</td>
<td>1.819 (0.034)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.899 (0.007)</td>
<td>0.902 (0.031)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.663 (0.025)</td>
<td>2.678 (0.039)</td>
</tr>
<tr>
<td>B.</td>
<td></td>
<td>2.441 (0.020)</td>
<td>2.420 (0.054)</td>
</tr>
</tbody>
</table>

Table 10. Throughput (TP) Estimation with $K = 20$

<table>
<thead>
<tr>
<th>Sys</th>
<th>Svr</th>
<th>Simulated TP</th>
<th>Estimated TP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0.0867 (0.0004)</td>
<td>0.0893 (0.0016)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0917 (0.0005)</td>
<td>0.0915 (0.0003)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0988 (0.0006)</td>
<td>0.1012 (0.0017)</td>
</tr>
<tr>
<td>A.</td>
<td></td>
<td>0.0942 (0.0005)</td>
<td>0.0931 (0.0009)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0839 (0.0003)</td>
<td>0.0855 (0.0006)</td>
</tr>
<tr>
<td>B.</td>
<td></td>
<td>0.1152 (0.0006)</td>
<td>0.1137 (0.0005)</td>
</tr>
</tbody>
</table>

$p(1, 2) = 0.6$, $p(1, 3) = 0.4$, $p(2, 1) = 0.3$, $p(2, 3) = 0.7$, $p(3, 1) = 0.6$, and $p(3, 2) = 0.4$. The routing probabilities of system B are $p(1, 2) = 0.4$, $p(1, 3) = 0.6$, and the others are the same as system A's. We applied Algorithm 4 to both systems A and B with $K = 20$ and $K = 50$. The simulation was repeated five times with different random seeds and $N = 100,000$ to obtain the means and standard deviations. The averages of the simulated and estimated throughputs (TP) and mean queue lengths (QL) are listed in Tables 8–11, with numbers in parentheses indicating the standard deviations. The simulated TP and QL of system A (or B) are obtained directly from simulation of system A (or B), while the estimated TP and QL of system A (or B) are obtained from applying Algorithm 4 to the simulation of system B (or A). For QL of server $i$, $i = 1, 2, 3$, the performance function if $f(n) = n$. For TP of server $i$, $TP_i$, we first estimate the utilization of server $i$, $U_i$, with the performance function $f(n) = e(n)$; and then let $TP_i = U_{bs} \cdot n$, $i = 1, 2, 3$.

The results indicate that the variances of QL are usually bigger than those of TP. This is not surprising because the performance function for QL is usually larger.

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than that of TP. The results also indicate that although it looks as if there are some biases for $K = 20$, the estimates are still reasonably accurate.

Example 4

In this example, we study a five-server closed network with finite buffer sizes. There are 10 customers circulating in the network. The service times are exponentially and independently distributed with means 5, 4, 8, 8, and 7, respectively. Each server has a finite buffer with a size of 5; thus, blocking may occur in this network. No product-form solution exists for this network. The routing probabilities of system A are the same as those listed in Table 4. The routing probabilities of system B are $p(1, 3) = 0.4$ and $p(1, 4) = 0.1$, with the rest being the same as those of system A. We applied Algorithm 4 to system B to estimate the throughputs and mean queue lengths of system A. The results are listed in Table 12, with $K = 200$.

6 EAIS VERSUS TAIS

From Equations (4) and (11), the likelihood ratio

$$G_n = \prod_{i=0}^{K-1} \frac{p'(X_i, X_{i+1})}{p(X_i, X_{i+1})}$$

plays an important role in both TAIS and EAIS. First let us study the property of $G_n$. We assume that $p'(i, j) = 0$ whenever $p(i, j) = 0$. It is easy to show that

$$E \left\{ \frac{\prod_{i=0}^{K-1} p'(X_i, X_{i+1})}{p(X_i, X_{i+1})} X_0, X_1, \ldots, X_i \right\} = 1.$$  \hspace{1cm} (21)

Recall that $\Sigma^N$ is the smallest $\sigma$-field on which the $N$-transition sample paths are measurable. We have $\Sigma^N \subset \Sigma^{N+1}$ and $E[|G_n|] < \infty$. From Equation (21), we have

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Table 12. Queue-Length (QL) and Throughput (TP) Estimation for a Five-Server System with \( K = 200 \)

<table>
<thead>
<tr>
<th>Svr</th>
<th>Simulated</th>
<th>Estimated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1049</td>
<td>0.1143 (0.0075)</td>
</tr>
<tr>
<td>2</td>
<td>0.0632</td>
<td>0.0656 (0.0038)</td>
</tr>
<tr>
<td>TP</td>
<td>3</td>
<td>0.0519 (0.0047)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.1058 (0.0029)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.1101 (0.0054)</td>
</tr>
<tr>
<td>1</td>
<td>1.635</td>
<td>1.652 (0.188)</td>
</tr>
<tr>
<td>2</td>
<td>0.465</td>
<td>0.454 (0.040)</td>
</tr>
<tr>
<td>QL</td>
<td>3</td>
<td>1.001 (0.081)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3.611 (0.012)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>3.327 (0.060)</td>
</tr>
</tbody>
</table>

\[
E(G_{n+1}, \Sigma^{(k)}) = E(G_{n}, X_0, \ldots, X_n) \\
= G_n E \left( \frac{p'(X_{n+1}, X_{n+2})}{p(X_{n+1}, X_{n+2})} \ \mid X_0, \ldots, X_n \right) = G_n.
\]

Thus, \( G_n \) is a martingale relative to \( \Sigma^{(k)} \), \( \Sigma^{(k)} \), \ldots \ (8). Then the martingale convergence theorem leads to the conclusion that, as \( N \to \infty \), \( G_n \) converges with probability one to a random variable \( G \). In fact, [2] shows that unless \( p'(i, j) = p(i, j) \) for all \( i, j \), we have

\[
\lim_{n \to \infty} G_n = 0 \quad w.p.1.
\]

(22)

This result is intuitively true. \( G_n \) represents the "likelihood" that an \( N \)-transition sample path of Markov chain \( X \) with transition probabilities \( p(i, j) \) is also a sample path of Markov chain \( X' \) with different transition probabilities \( p'(i, j) \). As \( N \) becomes larger, the possibility of such an event becomes smaller. As \( N \) goes to infinity, such a likelihood goes to zero.

Letting \( I = 0 \) in Equation (21), we get

\[
E(G_0) = 1 \quad \text{for all } N.
\]

(23)

This implies

\[
\lim_{n \to \infty} E(G_0) = 1.
\]

(24)
Equations (22) and (24) indicate some properties of $G_N$. $G_N$ is not uniformly integrable on $(Ω, Σ, P)$, or $G_N$ is not uniformly bounded. That is, despite Equation (22), for any big real number $R$, there is always an $N$ large enough such that some values of $G_N$ are larger than $R$. By Equation (22), as $N$ increases, $G_N$ approaches zero on more points (sample paths), and the values of $G_N$ on other points grow bigger to keep Equation (24) satisfied. This indicates that as $N$ increases, the values of $G_N$ spread wider and hence the variance of $G_N$ increases.

Now, let us study the variance of $G_N$. From Equation (21), we have

$$G_N = \exp(H_N),$$

(25)

where

$$H_N = \sum_{k=0}^{N-1} \phi(X_k, X_{k+1}) \quad \text{with} \quad \phi(X_k, X_{k+1}) = \ln \frac{p(X_k, X_{k+1})}{f(X_k, X_{k+1})}.$$

Since $\phi(X_k, X_{k+1})$'s are not independent, we need the central limit theorem with dependent variables. Using an augmented Markov chain with states $Y_n = (X_n, X_{n+1})$, we can apply the same method as in Examples 27.5 and 27.7 of [8] to show that $\{\phi(X_n, X_{n+1})\}$ is $\alpha$-mixing. Thus, $\phi(X_k, X_{k+1})$ and $\phi(X_{k+1}, X_{k+2})$ are approximately independent for large $k$. It can be easily verified that other conditions in Theorem 27.5 of [8] are satisfied and

$$\frac{H_N - N\bar{\phi}}{\sigma \sqrt{N}} \Rightarrow N(0, 1),$$

(26)

where $\bar{\phi} = \sum_{i,j} \pi(i)\pi(j)\phi(i, j)$ is the steady-state mean of $\phi(X_k, X_{k+1})$ and $N(0, 1)$ is the standard normal distribution. Equation (26) indicates that the variance of $H_N = \ln(G_N)$ is asymptotically proportional to $N$.

The preceding analysis shows clearly that EAIS estimates have much smaller variance than the TAIS ones, since the $X$ in Equation (11) can be chosen much smaller than the $N$ in Equation (4). In fact, to reduce the variance, the TAIS method has to employ the regenerative structure. From the basic regenerative theory, the steady-state performance can be expressed as a ratio. We choose $X = i$ as regenerative points. Let $\tau_0, \tau_1, \ldots$, be successive regenerative points, and without loss of generality, we assume that $\tau_0 = 0$. Then, the steady-state performance $E_s(f) = \Sigma_{i=0}^{\infty} f(i)\pi(i)$ is

$$E_s(f) = \frac{E[\Sigma_{i=0}^{\infty} f(X_i)]}{E(\tau_1)}.$$

Applying TAIS to both the numerator and denominator, we have the steady-state performance for $X'$, $E_s(f) = \Sigma_{i=0}^{\infty} f(i)\pi'(i)$, as follows

$$E_s(f) = \frac{E[\Sigma_{i=0}^{\infty} f(X_i)G_i]}{E(\tau_1G_1)}.$$
Note that $\tau$ is a stopping time, which is an unbounded random variable. Thus, the variance of $G_\tau$ is expected to be larger than that of $G_K$ for a finite $K$ in the EAIS method. In fact, it is proved in [2] that if $p(j, i) > 0$ for all $j \in \Phi$ and $\min_{i, j} \sum_{k} p_{ij}(j, k) > 1$, then the variance of $G_\tau$ is infinity. The EAIS method introduced herein does not have this problem.

7 CONCLUSION

Given a sequence of samples of a random variable $x, x_1, x_2, \ldots, x_m$, one can estimate the performance of any other random variable $y$ by applying the importance sampling technique. The application of this technique to stochastic systems means that given a set of sample paths of a stochastic system, one can estimate the performance measure of any other system with the same structure. However, the straightforward application (i.e., the TAIS method) of the technique to the steady-state performance measures of Markov processes usually leads to very big variances. The TAIS method updates the probability measure after a long simulation run to achieve the steady state, and the modifying factor depends on the entire long simulation period. Regenerative periods are often used to reduce the variance. Even with the regenerative structure, the variance of the TAIS estimate may be infinity.

In this article, we proposed a different approach called EAIS. In this approach, the steady-state performance is not directly measured. Instead, we estimate the steady-state probability $\pi(j), j = 1, 2, \ldots, M$, by estimating $P(X_{n+\tau} = j|X_n = i)$ by using ensemble average. Because $P(X_{n+\tau} = j|X_n = i)$ converges to $\pi(j)$ much faster than the long time average to the steady-state performance, $K$ can be chosen much smaller than the length of the simulation run in the TAIS case. The variance can be significantly reduced. Regenerative periods are unnecessary; hence the EAIS method is applicable to a wide range of systems.

The application of EAIS is twofold: First, with EAIS, one can estimate the performance of different systems by observing a single sample path of one system (which may be a real one). Second, it provides an efficient method to estimate the performance of many systems by simulating only one of them.

One important issue with the EAIS approach is how to choose an appropriate $K$ so that the bias and variance of the estimate are well balanced. An upper bound of the bias for a given $K$ is derived, and some qualitative discussion about the variance is given. These results provide us some heuristics based on which we can determine the size of $K$. A practically useful procedure for choosing $K$ is, however, to be developed.

REFERENCES


Xi-Ren Cao received his M.S. and Ph.D. degrees from Harvard University in 1981 and 1984, respectively, where he was a research fellow from 1984 to 1986. He then worked as a principal and consultant engineer at Digital Equipment Corporation, U.S.A., until October 1993. Since then, he has been a professor of the Hong Kong University of Science and Technology. He owns two patents in data communication and has published two books: Realization Probabilities—The Dynamics of Queuing Systems (Springer-Verlag, 1994), and Perturbation Analysis of Discrete-Event Dynamic Systems (Kluwer Academic Publishers, 1991), which was co-authored with Y.C. Ho. He received the Outstanding Transactions Paper Award from the IEEE Control System Society in 1987 and the Outstanding Publication Award from the Institution of Management Science in 1990. He served as associate editor of the IEEE Transactions on Automatic Control and the IEEE Transactions on Control Systems Technology. He is currently associate editor of the Journal of Discrete-Event Dynamic Systems and area editor of International Journal of Operations and Quantitative Management. He is a Fellow of IEEE and the chairman of the Technical Committee of Networks and Communications of IEEE Control Systems Society. His current research interests include discrete event systems, computer and communication networks, stochastic processes, and optimization techniques.