Generalized LQR Control and Kalman Filtering With Relations to Computations of Inner–Outer and Spectral Factorizations

Guoxiang Gu, Xi-Ren Cao, and Hesham Badr

Abstract—**We investigate the generalized linear quadratic regulator (LQR) control where the dimension of the control input is strictly greater than the dimension of the controlled output, and the weighting matrix on the control signal is singular. The dual problem is the generalized Kalman filtering where the dimension of the input noise process is strictly smaller than the dimension of the output measurement, and the covariance of the observation noise is singular. These two problems are intimately related to inner–outer factorizations for nonsquare stable transfer matrices with square inners of the smaller size. Such inner–outer factorizations are in turn related to spectral factorizations for power spectral density (PSD) matrices whose normal ranks are not full. We propose iterative algorithms and establish their convergence for inner–outer and spectral factorizations, which in turn solve the generalized LQR control and Kalman filtering.**

Index Terms—**Inners/outers, Kalman filtering, linear-quadratic control, spectral factorizations.**

I. Introduction

In THE standard linear quadratic regulator (LQR) control, the dimension of the control input is no greater than the dimension of the controlled output, and the weighting matrix on the control signal is nonsingular. For the standard Kalman filtering, the dimension of the input noise process is no smaller than the dimension of the output measurement, and the covariance of the observation noise is nonsingular. The standard LQR control and Kalman filtering are well studied, and their solutions and properties are well documented [2], [11]. It is interesting to observe that these two optimization problems are related to, and have applications to computations of inner–outer and spectral factorizations [1], [7].

In this paper, we study generalized LQR control and Kalman filtering for discrete-time systems in which the aforementioned regular conditions fail. While the solutions to these two generalized optimization problems can be obtained from similar Riccati equations to those in the regular case, it is not easy to compute the stabilizing solutions to the algebraic Riccati equations (AREs), associated with the generalized LQR control and Kalman filtering. Such AREs involve pseudoinverses, and are often referred to as generalized AREs. In fact for stable systems the two generalized optimization problems can be equivalently converted to inner–outer factorizations for nonsquare stable transfer matrices whose inners are square and have a smaller size that is in turn related to the spectral factorization for power spectral density (PSD) matrices whose normal ranks are not full. Our approach to the generalized LQR control and Kalman filtering is through tackling the equivalent inner–outer and spectral factorizations, from which we develop an iterative algorithm for computing the stabilizing solutions to the AREs associated with the two optimization problems. We will prove the convergence of the proposed iterative algorithm.

Spectral factorizations have been widely used in signal processing, control, and communications, due to the need for spectral analysis in signals and for frequency-domain design in systems. There is a large body of literatures devoted to such a topic [1], [6], [14], [18], [19]. The solutions given in [1], [19] are the most general, but both did not cover those PSD matrices whose normal ranks are not full. Such spectral factorizations are less studied, and much harder to compute. Nevertheless its solution helps to solve the generalized LQR control and Kalman filtering. In addition the blind channel estimation emerged in wireless data communications [5], [13], [15] is equivalent to such spectral factorizations. We will follow the state–space approach in [1], and develop convergent iterative algorithms to compute spectral factors for PSD matrices with nonfull normal ranks.

The stabilizing solutions to the generalized AREs have been studied (see [4], [12], and the references therein), which are applicable to the generalized LQR control and Kalman filtering. However the existing approach is based on the augmented matrix pencil via computing the stable deflating subspace. The difficulty and complexity in computing accurate deflating subspaces render the numerical solutions to the generalized AREs less reliable than in the regular case [4, p. 168]. In addition the AREs associated with the generalized LQR control and Kalman filtering do not cover those AREs for spectral factorization considered in this paper. Our contribution is that we not only propose two iterative algorithms for computing solutions to the generalized LQR control and Kalman filtering, and to the spectral factorization, respectively, but also show the relation between these two iterative algorithms which ultimately leads to the proof of the convergence of the proposed iterative algorithms to their respective stabilizing solutions.

The contents of this paper are organized as follows. After the introduction section, the problems of inner–outer and spectral

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factorizations are formulated in Section II where the mathematical notations are introduced. Section III is devoted to the generalized LQR control and Kalman filtering, and proposes the iterative algorithm for computing the positive–semidefinite solutions to the corresponding AREs. The convergence of the iterative algorithm is studied in Section IV via another iterative algorithm for spectral factorizations in which PSD matrices do not have full normal ranks. The relation of these two iterative algorithms is discovered with their convergence property proven. The paper is concluded in Section V with illustrative examples and remarks.

II. PRELIMINARIES

We will begin with the formulation of the inner–outer and spectral factorizations entailed in this paper. Denote the set of real/complex numbers by $\mathbb{F} = \mathbb{R}/\mathbb{C}$. Let $H(z)$ be a transfer function matrix of size $p \times m$. It is called causal, if its impulse response is causal. Its normal rank is defined as the rank of $H(z)$ for almost all, except a countable set of $z \in \mathbb{C}$. Denote $\bar{a}$ the conjugate of a, and A^* the conjugate and transpose of A. Then the para-hermitian conjugate of $H(z)$ is defined and denoted by $H(z)^\sim = [H(\bar{z}^{-1})]^*$. Assume that $H(z)$ admits a state–space realization

$$H(z) = D + C(zI - A)^{-1}B =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad (1)$$

by an abuse of notation, where $A \in \mathbb{F}^{n\times n}$ and $D \in \mathbb{F}^{p\times m}$. It is clear that B and C have dimensions of $n \times m$ and $p \times n$, respectively. If A is a stability matrix, i.e., all eigenvalues of A are strictly inside the unit circle, then $H(z)$ is stable. If

$$\text{rank}\left\{\left[\begin{array}{c|c} A - zI & B \\ \hline C & D \end{array}\right]\right\} = n + \min\{p, m\}\ \forall\, |z| \le 1 \quad (2)$$

then $H(z)$ is strict minimum phase. If the previous rank condition holds for only $|z| < 1$, then $H(z)$ is minimum phase. Notice that neither ensures the full rank of D.

A para-hermitian matrix $\Phi(z)$ of size $p \times p$ has the form

$$\Phi(z) = \sum_{k=-\infty}^{\infty} R_k z^{-k},\ R_k^* = R_{-k} \in \mathbb{F}^{p\times p}. \quad (3)$$

It follows that $\Phi(z)$ is a hermitian matrix for any z on the unit circle. If in addition $\Phi(z) \ge 0\ \forall\, |z| = 1$, then $\Phi(z)$ qualifies a PSD with $\{R_k\}$ the covariance sequence. Let the normal rank of $\Phi(z)$ be $r < p$. We are interested in spectral factorizations

$$\Phi(z) = W_R(z)^\sim W_R(z) = W_L(z) W_L(z)^\sim \quad (4)$$

where $W_R(z)$ has size $r \times p$, $W_L(z)$ has size $p \times r$, and more importantly both are causal, stable, and strict minimum phase. In other words, all poles and zeros of $W_R(z)$ and $W_L(z)$ are strictly inside the unit circle. In this case $W_R(z)$ and $W_L(z)$ are called right and left spectral factors of $\Phi(z)$. Spectral factors are unique upto a factor of unitary matrices. Extensions can be made for spectral factors to include poles and zeros on the unit circle. However, for the sake of simplicity and brevity, we shall not do so in this paper. Instead we assume that $\Phi(z)$ is a bounded hermitian positive matrix with rank r for all z on the unit circle, which excludes the possibilities of poles and zeros on the unit circle for the spectral factors. It is worth to pointing out that most of the existing work on spectral factorizations assume that $r = p$, and there lack effective computational algorithms for spectral factorizations in the case of $0 < r < p$.

In this paper, we will also consider more general inner–outer factorizations where $H(z)$ as given in (1) may have zeros strictly outside the unit circle, and its realization is subject to the constraint

$$0 < \text{rank}\{D\} \le \min\{m, p\}. \quad (5)$$

We investigate inner–outer factorizations for the following two cases:

$$\begin{aligned} &\text{Case i) } m > p: \quad H(z) = H_{\rm i}(z) H_{\rm o}(z) \\ &\text{Case ii) } m < p: \quad H(z) = H_{\rm o}(z) H_{\rm i}(z) \end{aligned} \quad (6)$$

where $H_{\rm i}(z)$ is a square inner of the smaller size, and $H_{\rm o}(z)$ is an outer. A square transfer matrix $H_{\rm i}(z)$ is called inner, if it is stable, and $H_{\rm i}(e^{j\omega})$ is a unitary matrix for all $\omega \in \mathbb{R}$. In other words, $H_{\rm i}(z)^\sim H_{\rm i}(z) = H_{\rm i}(z) H_{\rm i}(z)^\sim = I$. A nonsquare transfer matrix $H_{\rm o}(z)$ is called outer, if it is both stable, and minimum phase. A moment of reflection reveals that all zeros of $H_{\rm i}(z)$ are strictly outside the unit circle, and are thus unstable. On the other hand, zeros of $H_{\rm o}(z)$ are all in the unit disc, including the unit circle. The aforementioned inner–outer factorizations are intimately related to spectral factorizations. In fact, $H_{\rm o}(z)$ is the right spectral factor of $\Phi(z) = H(z)^\sim H(z)$ for Case i), and is the left spectral factor of $\Phi(z) = H(z)H(z)^\sim$ for Case ii). The assumption that $D \ne 0$ has no loss of generality, because any causal transfer matrix $H(z)$ can be written as $H(z) = z^{-k}\tilde{H}(z)$ for some $k \ge 0$ and some causal transfer matrix $\tilde{H}(z)$ such that $\tilde{D} = \tilde{H}(\infty) \ne 0$. Thus, inner–outer factorizations of $\tilde{H}(z)$ can then be studied with z^{-k} subsumed into the inner.

It will be shown that the inner–outer factorizations in this paper have a close relation to the generalized LQR control and Kalman filtering. These two different problems amount to solving the stabilizing solutions of certain AREs. Due to the hypotheses on the generalized LQR control and Kalman filtering, the notion of pseudoinverses is needed. For a matrix $M \ne 0$, its pseudoinverse, denoted by M^+, satisfies $MM^+M = M$. More than one pseudoinverses exist in general. Let $M = USV^*$ be the singular value decomposition (SVD). One of the pseudoinverses of M is $M^+ = VS^+U^*$ where S^+ computes inverses of the nonzero diagonal elements of S.

III. GENERALIZED LQR CONTROL AND KALMAN FILTERING

In this section, we assume that the regular conditions for LQR control and Kalman filtering fail to hold, and investigate their optimal solutions. We propose an iterative algorithm to compute the optimal solutions. Its convergence to the stabilizing solution will be proven in the next section. We will also investigate inner–outer factorizations for nonsquare transfer matrices. For the interest of this paper, we restrict inners to square, and outers to nonsquare transfer matrices. Such inner–outer factorizations are less studied, let alone the singular constraint in (5). In the next two subsections, we review the results on optimal control and optimal estimation, investigate various properties associated with the generalized LQR control and Kalman filtering, and develop an iterative algorithm for computing inner–outer factorizations with square inners.

A. Generalized LQR Control and Right Spectral Factor

The generalized LQR control assumes the state–space model

$$x(t+1) = Ax(t) + Bu(t) \quad z(t) = Cx(t) + Du(t) \tag{7}$$

with $x(0) = x_0 \neq 0$. It searches for the control input $u(t)$ to minimize the quadratic performance index

$$J = \sum_{t=0}^{\infty} \|z(t)\|^2 = \sum_{t=0}^{\infty} z^*(t)z(t). \tag{8}$$

We assume that the control input $u(t)$ has size m, the controlled output $z(t)$ has size p, and $m > p$. Stability of A is not assumed for the generalized LQR control, and $\text{rank}\{D\} \leq p$.

This problem differs from the standard LQR problem in that D is a "fat" matrix by $m > p$, and its rank can be strictly smaller than p. That is, the penalty weighting matrix on the control signal is singular regardless of the rank of D. The following result is adapted from the existing literature.

Theorem 3.1: Let the m-input/p-output system be given as in (7) with $m > p \geq \text{rank}\{D\}$. Suppose that (A, B) is stabilizable, and $X = X^* \geq 0$ satisfies the generalized ARE

$$X = A^*XA + C^*C - S^*(D^*D + B^*XB)^+S \tag{9}$$

where $S = B^*XA + D^*C$. Then with $u_{\text{opt}}(t) = Fx(t)$

$$F = -(D^*D + B^*XB)^+(B^*XA + D^*C) \tag{10}$$

the performance index is $J = J_{\min} = x_0^*Xx_0$.

The optimal solution in the above theorem can be derived in a similar fashion to that for the standard LQR control using finite horizon optimal control. Indeed the ARE in (9) can be obtained by taking the time limit in the following difference Riccati equation (DRE):

$$\begin{aligned} X_t &= A^*X_{t+1}A + C^*C - S_{t+1}^*(D^*D + B^*X_{t+1}B)^+S_{t+1} \\ &= \overline{A}_t^*X_{t+1}\overline{A}_t + (C + DF_t)^*(C + DF_t) \end{aligned} \tag{11}$$

for some $X_T \geq 0$ and with the feedback gain

$$F_t = -(D_t^*D_t + B_t^*X_{t+1}B_t)^+(B_t^*X_{t+1}A_t + D_t^*C_t)$$

where $\overline{A}_t = A + BF_t$ and $S_{t+1} = B^*X_{t+1}A + D^*C$. It can be shown that with $u(t) = F_tx(t)$ for all t, the finite horizon performance index

$$J_T = x^*(T)X_Tx(T) + \sum_{t=0}^{T-1} \|z(t)\|^2 \tag{12}$$

is minimized. In fact the finite horizon LQR control is applicable to time-varying systems as well.

The above discussion suggests the iterative algorithm: For $k = 0, 1, \ldots$, do the following:

$$\begin{aligned} F^{(k)} &= -(D^*D + B^*X^{(k)}B)^+(B^*X^{(k)}A + D^*C) \\ X^{(k+1)} &= \overline{A}_{(k)}^*X^{(k)}\overline{A}_{(k)} + \overline{C}_{(k)}^*X^{(k)}\overline{C}_{(k)} \end{aligned} \tag{13}$$

where the initial value $X^{(0)} \geq 0$, $\overline{A}_{(k)} = A + BF^{(k)}$, and $\overline{C}_{(k)} = C + DF^{(k)}$. The algorithm can be terminated if $\|X^{(N)} - X^{(N+1)}\|$ is smaller than some pre-specified tolerance bound. It is noted that $X^{(k)} = X_{T-k}$ is the solution to the DRE in (11) at time $t = T - k$ with the boundary condition $X_T = X^{(0)} \geq 0$. If $t = T - k \geq 0$ is finite and fixed as $T > k \to \infty$, then $X^{(k)} \to X_\infty(t) = X$. Hence, the iterative algorithm in (13) is convergent for any initial value $X^{(0)} \geq 0$.

As in the standard LQR theory, we can not conclude stability of $(A + BF)$ despite the fact that (A, B) is stabilizable. That is, the optimal feedback system

$$x(t+1) = (A + BF)x(t) \quad z(t) = (C + DF)x(t) \tag{14}$$

with F in (10), may not be internally stable, even though the energy of the controlled output

$$J_{\min} = \|z\|_2^2 = \sum_{t=0}^{\infty} \|z(t)\|^2 = x_0^*Xx_0$$

is bounded. A careful reflection concludes that any unstable modes of $(A + BF)$ are unobservable based on the controlled output $z(t) = (C + DF)x(t)$. That is, the unstable modes of $(A + BF)$ are also unobservable modes of $(C + DF, A + BF)$.

Remark 3.2: We make the following remarks.

a) The ARE (9) may admit more than one positive–semidefinite solutions. Each one can be viewed as an equilibrium to the DRE in (11). However, there is a unique maximal solution $X_{\max}$, and a unique minimal solution $X_{\min}$ such that any other positive–semidefinite solution X to the ARE (9) satisfies $0 \leq X_{\min} \leq X \leq X_{\max}$. If the initial value $X^{(0)} = 0$ for the iterative algorithm in (13), then $X^{(k)}$ is likely to converge to $X_{\min}$ as $k \to \infty$. This is intuitively true based on the optimality of the feedback control gain corresponding to $X^{(t)} = X_{T-t}$ for each $t \geq 0$. On the other hand, if $X^{(0)} = \rho I$ with $\rho > 0$ sufficiently large, then $X^{(k)}$ is likely to converge to $X_{\max}$ as $k \to \infty$. In particular, if $X^{(0)} \geq 0$ is close to some $X \geq 0$ satisfying the ARE (9), then $X^{(k)}$ is likely to be trapped to the same X in a few iterations.

b) For the problem of inner–outer factorizations in Case (i) of (6), A is assumed to be a stability matrix. If the initial value $X^{(0)}$ satisfies the Lyapunov equation

$$X^{(0)} = A^*X^{(0)}A + C^*C \tag{15}$$

then $X^{(0)} \geq 0$. Moreover, taking the difference between (15) and (9) yields

$$(X^{(0)} - X) = A^*(X^{(0)} - X)A + S^*(D^*D + B^*XB)^+S.$$

Stability of A implies that $X^{(0)} \geq X$ for any positive–semidefinite solution to the ARE (9). Hence, the maximal solution to the ARE (9) is likely to be obtained with the iterative algorithm (13) using the solution to (15) as the initial value. □

A solution $X \geq 0$ to the ARE (9) is said to be a stabilizing solution, if $(A+BF)$ is a stability matrix where F has the expression in (10). Similar to the regular case it can be shown that $X = X_{\max}$ is the stabilizing solution, and its existence is hinged to stabilizability of (A, B), and

$$\operatorname{rank}\left\{\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}\right\} = n + p \qquad \forall\, \theta \in \mathbb{R}. \tag{16}$$

For ease of the reference, we denote F_{m} as the optimal stabilizing feedback gain given by

$$F_{\mathrm{m}} = -(D^*D + B^*X_{\max}B)^{+}(B^*X_{\max}A + D^*C). \tag{17}$$

In the rest of this section, we will examine inner–outer factorization for Case i) in (6).

Let $X \geq 0$ be a solution to the ARE (9), and the state feedback gain F be as in (10). Denote $\Pi = D^*D + B^*XB$. Then

$$\Pi F = -S = -(B^*XA + D^*C). \tag{18}$$

Indeed for any matrix W there holds identity[1] $W^*WW^+ = W^*$. Set

$$W = \begin{bmatrix} X^{1/2}B \\ D \end{bmatrix} \quad Z = \begin{bmatrix} X^{1/2}A \\ C \end{bmatrix}$$

yielding $F = -(W^*W)^+W^*Z$. Thus

$$\begin{aligned} \Pi F &= W^*WF = W^*WW^+[W^*]^TW^*Z \\ &= W^*[W^*]^TW^*Z = -W^*Z = -S \end{aligned}$$

that verifies (18). The next lemma shows that $\Pi = D^*D + B^*XB$ has the same rank as the normal rank of $H(z)$.

Lemma 3.3: Suppose that $\operatorname{rank}\{D\} \leq p < m$ with $p \times m$ the dimension of $H(z)$ as in (1). Let $X \geq 0$ be a solution to the ARE (9), and the state feedback gain F be as in (10). Denote $\Theta(z) = (zI - A)^{-1}B$ and $\Pi = D^*D + B^*XB$. Then there holds

$$H(z)^{\sim}H(z) = [I - F\Theta(z)]^{\sim}\,\Pi\,[I - F\Theta(z)]. \tag{19}$$

Proof: By the property of pseudoinverses, the ARE (9) can be written as

$$X - A^*XA = -F^*(D^*D + B^*XB)F + C^*C. \tag{20}$$

The verification of the result in (19) is similar to that in the regular case, and is thus omitted. □

Since $[I - F\Theta(z)]$ has the full normal rank, the rank of $\Pi = (D^*D + B^*XB)$ is the same as the normal rank of $H(z)$ that is a useful property. The following observation is also important.

Lemma 3.4: Suppose that $H(z)$ of size $p \times m$ as in (1) has normal rank p, and $\operatorname{rank}\{D\} \leq p < m$. Let $X \geq 0$ be a solution to the ARE (9), and the state feedback gain F be as in (10). Let $\Pi = \Omega^*\Omega$ be its Cholesky factorization with $p \times m$ the size of Ω, and Ω^+ of size $(m-p) \times m$ satisfy

$$\Omega_\perp^+\Omega_\perp = I - \Omega^+\Omega \quad \det\left(\begin{bmatrix} \Omega \\ \Omega_\perp \end{bmatrix}\right) \neq 0. \tag{21}$$

Such an Ω^+ has rank $(m-p)$. Then

$$G(z) = (C + DF)(zI - A - BF)^{-1}B(I - \Omega^+\Omega) = 0. \tag{22}$$

Proof: The identity (19) shows that the rank of $\Pi = D^*D + B^*XB$ is the same as the normal rank of $H(z)$, which is p. Hence, there exist Ω of size $p \times m$ and $\Omega_\perp$ of size $(m-p) \times m$ such that $\Pi = \Omega^*\Omega$, and (21) holds. With the previous notation, the feedback gain in (10) has the expression $F = -\Omega^+(\Omega^+)^*(B^*XA + D^*C)$. It follows that

$$(I - \Omega^+\Omega)F = -\Omega_\perp^+\Omega_\perp\Omega^+(\Omega^+)^*(B^*XA + D^*C) = 0 \tag{23}$$

by $\Omega_\perp\Omega^+ = 0$ as in (21). Let the rank of X be $r > 0$. Then

$$X = [\,U_1 \quad U_2\,]\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = U_1\Sigma_1U_1^* \tag{24}$$

with $\Sigma_1 = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) > 0$ by the SVD of X where $U = [\,U_1 \quad U_2\,]$ is a unitary matrix. Then

$$\begin{aligned} \Pi = \Omega^*\Omega &= B^*XB + D^*D \\ &= [\,B^*U_1\Sigma^{1/2} \quad D^*\,]\begin{bmatrix} \Sigma_1^{1/2}U_1^*B \\ D \end{bmatrix}. \end{aligned}$$

Thus, $\Omega\Omega_\perp^+ = 0$ implies that $D\Omega_\perp^+ = 0$, and $U_1^*B\Omega_\perp^+ = 0$, yielding

$$\begin{aligned} D\Omega_\perp^+\Omega_\perp &= D(I - \Omega^+\Omega) = 0 \\ U_1^*B\Omega_\perp^+\Omega_\perp &= U_1^*B(I - \Omega^+\Omega) = 0. \end{aligned} \tag{25}$$

In light of the expression in (20), the ARE (9) can be rewritten into the form of the following Lyapunov equation:

$$X = (A + BF)^*X(A + BF) + (C + DF)^*(C + DF). \tag{26}$$

Clearly, all unstable poles of $G(z)$ as in (22) are unobservable modes of $(C+DF, A+BF)$. Applying the similarity transform $S = U = [\,U_1 \quad U_2\,]$ to the realization of $G(z)$ gives

$$\begin{aligned} U^*\overline{A}U &= \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}(A + BF)[\,U_1 \quad U_2\,] \\ &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ U^*B\Omega_\perp^+\Omega_\perp &= \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}B(I - \Omega^+\Omega) = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \overline{C}U &= (C + DF)[\,U_1 \quad U_2\,] = [\,C_1 \quad C_2\,] \end{aligned}$$

with compatible partitions. It follows from (25) that $B_1 = 0$. Multiplying (26) by U^* from left, and U from right, and using the aforementioned partitions and the SVD of X in (24) yield

$$\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\Sigma_1[\,A_{11} \quad A_{12}\,] + \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix}[\,C_1 \quad C_2\,].$$

The fact that $\Sigma_1 > 0$ implies that $A_{12} = 0$ and $C_2 = 0$ in light of the previous equation, which coupled with $B_1 = 0$ concludes that (22) is true. □

[1]The authors thank one of the anonymous reviewers for bringing this identity to their attention.

We are now ready to present our result on the inner–outer factorization as in Case i) of (6).

Theorem 3.5: Suppose that $H(z)$ of size $p \times m$ as in (1) has normal rank $p < m$, satisfies the condition (16), and A is a stability matrix. Let $X_{\max} \geq 0$ be the maximal solution to (9), and $F_{\rm m}$ be as in (17). Then, there holds the inner–outer factorization $H(z) = H_{\rm i}(z)H_{\rm o}(z)$ where, with $\Omega_{\rm m}^*\Omega_{\rm m} = \Pi = D^*D + B^*X_{\max}B$, the inner and outer are given, respectively, by

$$\begin{aligned} H_{\rm i}(z) &= \left[\begin{array}{c|c} A+BF_{\rm m} & B \\ \hline C+DF_{\rm m} & D \end{array}\right]\Omega_{\rm m}^+ \\ H_{\rm o}(z) &= \Omega_{\rm m}\left[\begin{array}{c|c} A & B \\ \hline -F_{\rm m} & I \end{array}\right]. \end{aligned} \tag{27}$$

Proof: We note that the previous two lemmas hold for any solution $X \geq 0$ to the ARE (9). Thus, by the proof of Lemma 3.3 and $\Theta(z) = (zI - A)^{-1}B$

$$T(z) = H(z)[I - F_{\rm m}\Theta(z)]^{-1} = \left[\begin{array}{c|c} A+BF_{\rm m} & B \\ \hline C+DF_{\rm m} & D \end{array}\right]$$

satisfies $T(z)^\sim T(z) = D^*D + B^*X_{\max}B = \Omega_{\rm m}^*\Omega_{\rm m}$. That is, $\Omega_{\rm m}$ has the same column rank as the normal rank of $H(z)$, which is p, and $H_{\rm i}(z) = T(z)\Omega_{\rm m}^+$ is square and satisfies $H_{\rm i}(z)^\sim H_{\rm i}(z) = I_p$. Because the unstable modes of $(A+BF_{\rm m})$ are unobservable modes of $(C + DF_{\rm m}, A + BF_{\rm m})$, they can be eliminated through Kalman decomposition. Hence $H_{\rm i}(z)$ with minimal realization is stable, which is indeed an inner. To verify the expression of $H_{\rm o}(z)$ as in (27), we have

$$\begin{aligned} H_{\rm i}(z)H_{\rm o}(z) &= \left[\begin{array}{c|c} A+BF_{\rm m} & B \\ \hline C+DF_{\rm m} & D \end{array}\right]\Omega_{\rm m}^+\Omega_{\rm m}\left[\begin{array}{c|c} A & B \\ \hline -F_{\rm m} & I \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ -BF_{\rm m} & A+BF_{\rm m} & B\Omega_{\rm m}^+\Omega_{\rm m} \\ \hline -DF_{\rm m} & C+DF_{\rm m} & D \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A+BF_{\rm m} & -B(I-\Omega_{\rm m}^+\Omega_{\rm m}) \\ \hline C & C+DF_{\rm m} & D \end{array}\right] \\ &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = H(z) \end{aligned}$$

in light of Lemma 3.4. Stability of A ensures stability of $H_{\rm o}(z)$. Moreover the hypotheses of the theorem imply that the maximal solution $X_{\max}$ is stabilizing, or $(A+BF_{\rm m})$ is a stability matrix. Consequently, $H_{\rm o}(z)$ admits a right inverse given by

$$H_{\rm o}^+(z) = \left[\begin{array}{c|c} A+BF_{\rm m} & B \\ \hline F_{\rm m} & I \end{array}\right]\Omega_{\rm m}^+$$

which is stable. Hence, $H_{\rm o}(z)$ is strict minimum phase, and an outer. □

We comment that the outer factor $H_{\rm o}(z)$ has no transmission zeros at $z = \infty$, due to the full rank of $\Omega_{\rm m}$ which has size $p \times m$, and the same rank as the normal rank of $H(z)$. The possible transmission zeros of $H(z)$ at $z = \infty$ are now transmission zeros of the inner factor $H_{\rm i}(z)$, which is evident by its expression in (27). In the case when $H(z)$ is strict minimum phase (i.e., (2) is satisfied), there is a unique positive–semidefinite solution $X \geq 0$ to the ARE (9). In fact $X = 0$, if $\operatorname{rank}\{D\} = p$.

B. Generalized Kalman Filtering and Left Spectral Factor

The results in this subsection are dual to those in the previous subsection. Thus, we will only state the results without proofs and derivations. We will present the result on generalized Kalman filtering, and the inner–outer factorization for Case (ii) in (6), which assumes $p > m$. Since $H(z)H(z)^\sim = H_{\rm o}(z)H_{\rm o}(z)^\sim$, we seek a left spectral factor of $H(z)H(z)^\sim$, which is related to the generalized Kalman filtering. That is, we are given the random process described by

$$x(t+1) = Ax(t) + Bv(t) \quad y(t) = Cx(t) + Dv(t) \tag{28}$$

where $v(t)$ is a wide-sense stationary (WSS) random process, and satisfies

$$E[v(t)] = 0 \quad E[v(t+k)v^*(t)] = \delta(k)I \tag{29}$$

with $E[\cdot]$ the expectation operator and

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0. \end{cases}$$

The dimension of the input noise $\{v(t)\}$ is m, and the dimension of the output measurement $\{y(t)\}$ is p. Since $p > m$, the covariance of the observation noise $Dv(t)$ is singular. The objective is to estimate $x(t+1)$, based on the observation $\{y(k)\}_{k=0}^t$. The standard Kalman filtering deals with the case when D is "fat" and has the full-row rank. However, we have a "tall" D, which may not have a full-column rank: $0 < \operatorname{rank}\{D\} \leq m$.

By duality, assume that (C, A) is detectable. Let $Y = Y^* \geq 0$ be a solution to the ARE

$$\begin{aligned} Y &= AYA^* - S_Y(DD^* + CYC^*)^+S_Y^* + BB^* \quad (30) \\ &= (A+LC)Y(A+LC)^* + (B+LD)(B+LD)^* \\ L &= -(AYC^* + BD^*)(DD^* + CYC^*)^+ \quad (31) \end{aligned}$$

where $S_Y = (AYC^* + BD^*)$. Again there are more than one solution $Y \geq 0$ in general. A positive–semidefinite solution $Y \geq 0$ can be obtained iteratively: For $k = 0, 1, \ldots$, with $Y_0 \geq 0$, do the following:

$$\begin{aligned} L_k &= -(AY_kC^* + BD^*)(DD^* + CY_kC^*)^+ \\ Y_{k+1} &= (A+L_kC)Y_k(A+LC_k)^* + \overline{B}_{(k)}\overline{B}_{(k)}^* \end{aligned} \tag{32}$$

where $\overline{B}_{(k)} = B + L_kD$. In practice the algorithm is terminated when $\|Y_N - Y_{N+1}\|$ is smaller than some tolerance bound.

A dual result to Theorem 3.1 is that with estimator

$$\hat{x}(t+1) = (A+LC)\hat{x}(t) - Ly(t)$$

where L is as in (31), the error variance for state estimation

$$E[e^*(t)e(t)] = \text{Trace}\{E[e(t)e^*(t)]\} = \text{Trace}\{Y\}$$

is minimized. However $(A + LC)$ may not be stable, even though $Y \geq 0$ is a solution to the ARE (30). If $(A + LC)$ is unstable, then $(A + LC, B + LD)$ is an unreachable pair, and all unstable modes of $(A + LC)$ are unreachable modes of

$(A+LC, B+LD)$, by noting that the ARE (30) can be written into the form of Lyapunov equation

$$Y=(A+LC)Y(A+LC)^*+(B+LD)(B+LD)^*. \quad (33)$$

Moreover there are more than one positive–semidefinite solutions to (32), with only one $Y_{\max}$ and one $Y_{\min}$. Any other $Y \geq 0$ satisfies the inequality $Y_{\max} \geq Y \geq Y_{\min} \geq 0$. If in addition

$$\operatorname{rank}\left\{\begin{bmatrix} A-e^{j\theta}I & B \\ C & D \end{bmatrix}\right\} = n+m \qquad \forall\, \theta \in \mathbb{R} \quad (34)$$

then $Y_{\max}$ is stabilizing in the sense that with

$$L_{\mathrm{m}} = -(AY_{\max}C^*+BD^*)(DD^*+CY_{\max}C^*)^+ \quad (35)$$

$(A+L_{\mathrm{m}}C)$ is a stability matrix. As in the previous subsection, $Y_{\max}$ and L_{m} are associated with the inner–outer factorization entailed in Case ii) of (6).

Let $Y \geq 0$ be a solution to the ARE (30) and L as in (31). With $\Pi = DD^*+CYC^*$, there hold $L\Pi = -S_Y = -(AYC^*+BD^*)$, that is dual to (18), and

$$H(z)H(z)^\sim = [I-\Theta(z)L]\,\Pi\,[I-\Theta(z)L]^\sim \quad (36)$$

that is dual to Lemma 3.3 where $\Theta(z)=C(zI-A)^{-1}$. Moreover all unstable models of $A+LC$ are unreachable modes of $(A+LC, B+LD)$. The equality (36) shows that the rank of $\Pi = DD^*+CYC^*$ is the same as the normal rank of $H(z)$ in (1), which is parallel to the result in Lemma 3.3. If $H(z)$ as in (1) has normal rank $m<p$, then $\Pi = DD^*+CYC^*$ has rank m. The result parallel to Lemma 3.4 is

$$G(z)=(I-\Omega\Omega^+)C(zI-A-LC)^{-1}(B+LD)=0 \quad (37)$$

where $\Pi=\Omega\Omega^*$ is the Cholesky factorization with $p\times m$ the size of Ω. The next result presents the solution to the inner–outer factorization in Case ii) of (6).

Theorem 3.6: Suppose that $H(z)$ as in (1) has normal rank $m<p$, satisfies the condition (34), and A is a stability matrix. Let $Y=Y_{\max}\geq 0$ be the maximal solution to (32), and L_{m} be as in (35). Then there holds the inner–outer factorization $H(z)=H_{\mathrm{o}}(z)H_{\mathrm{i}}(z)$ where, with $\Omega_{\mathrm{m}}\Omega_{\mathrm{m}}^*=\Pi=DD^*+CY_{\max}C^*$, the inner and outer factors of $H(z)$ are given, respectively, by

$$H_{\mathrm{i}}(z)=\Omega_{\mathrm{m}}^+\left[\begin{array}{c|c} A+L_{\mathrm{m}}C & B+L_{\mathrm{m}}D \\ \hline C & D \end{array}\right]$$
$$H_{\mathrm{o}}(z)=\left[\begin{array}{c|c} A & -L_{\mathrm{m}} \\ \hline C & I \end{array}\right]\Omega_{\mathrm{m}}. \quad (38)$$

Although iterative algorithms are derived for computing solutions to the AREs in (9) and (30), it is unclear how to choose the initial value $X^{(0)}\geq 0$ and $Y_0 \geq 0$ to (13) and (32), respectively, that will ensure their convergence to the required stabilizing solutions. It turns out that such an issue has to be resolved together with that for spectral factorizations.

IV. Spectral Factorizations

In this section, we investigate the spectral factorization problem for the $q\times q$ para-hermitian matrix $\Phi(z)$ which is positive semidefinite on the unit circle. So $\Phi(z)$ is a PSD function and has the form

$$\Phi(z)=\sum_{k=-\infty}^{\infty} R_k z^{-k}$$
$$=R_0+C_\Phi(zI-A)^{-1}B_\Phi+B_\Phi^*(z^{-1}I-A^*)^{-1}C_\Phi^* \quad (39)$$

where A is a stability matrix, and the normal rank of $\Phi(z)$ is $\rho<q$. This problem is much harder than the case of full normal rank. Since $\Phi(z)\geq 0\ \forall |z|=1$, there exist factorizations [1]

$$\Phi(z)=W_G(z)^\sim W_G(z)=W_K(z)W_K(z)^\sim \quad (40)$$

where $W_G(z)$ of size $\rho\times q$ and $W_K(z)$ of size $q\times\rho$ are both stable, given by

$$W_G(z)=\left[\begin{array}{c|c} A & B_\Phi \\ \hline G & D_G \end{array}\right] \quad W_K(z)=\left[\begin{array}{c|c} A & K \\ \hline C_\Phi & D_K \end{array}\right] \quad (41)$$

for some (G, D_G) and (K, D_K). Such factorizations are referred to as *minimal degree factorizations* due to the same degrees of $W_G(z)/W_K(z)$ as the causal/anticausal part of $\Phi(z)$ in (39), which can not be made smaller without changing realization of $\Phi(z)$. The following result is translated from [1, p. 495] (positive real lemma for continuous-time systems).

Lemma 4.1: Suppose that $\Phi(z)\geq 0\ \forall|z|=1$, where A is a stability matrix. There exist minimal degree factorizations as in (40) for some $W_G(z)$ and $W_K(z)$ in the form of (41), if and only if

$$P=A^*PA+G^*G \quad C_\Phi=D_G^*G+B_\Phi^*PA \quad (42)$$
$$Q=AQA^*+KK^* \quad B_\Phi^*=D_KK^*+C_\Phi QA^* \quad (43)$$
$$R_0=D_G^*D_G+B_\Phi^*PB_\Phi=D_KD_K^*+C_\Phi QC_\Phi^* \quad (44)$$

admit solutions (P, G, D_G), and (Q, K, D_K), respectively.

Lemma 4.1 shows that in order to obtain the minimal degree factors $W_G(z)$ and $W_K(z)$ in (41), i.e., (G, D_G) and (K, D_K), we need first solve for P and Q in (42) and (43), respectively, which are two Lyapunov equations. Since A is stable, $P\geq 0$ and $Q\geq 0$, if they exist. However, more than one set of such solutions (P, G, D_G), or (Q, K, D_K) exist, implying that more than one pair of minimal degree factors exist. However, there are unique sets of solutions (P, G, D_G) and (Q, K, D_K) such that both $W_G(z)$ and $W_K(z)$ as in (41) are outer functions, i.e.,

$$\operatorname{rank}\{W_G(z)\}=\operatorname{rank}\{W_K(z)\}=\rho \qquad \forall\,|z|\geq 1. \quad (45)$$

Such $W_K(z)$ and $W_G(z)$ are exactly the left and right spectral factors, respectively.

The spectral factorization problem in this section is also referred to as minimal degree spectral factorizations [1], and the spectral factors are unique up to a factor of unitary matrices. Because not every set of solutions to (42) or to (43) yields spectral factors of $\Phi(z)$, our goal is to obtain the right sets of solutions such that the resultant $W_G(z)$, and $W_K(z)$ are spectral factors of $\Phi(z)$, and satisfy (40). For this purpose, the results from the previous section play the pivotal role.

In light of (42) and (43), $D_G^*G=C_\Phi-B_\Phi^*PA$ and $D_KK^*=B_\Phi^*-C_\Phi QA^*$, implying that

$$\mathcal{R}(C_\Phi-B_\Phi^*PA)\subseteq\mathcal{R}(D_G^*) \quad \mathcal{R}(B_\Phi^*-C_\Phi QA^*)\subseteq\mathcal{R}(D_K).$$

Thus, $G=(D_G^+)^*(C_\Phi-B_\Phi^*PA)$, $K=(B_\Phi-AQC_\Phi^*)(D_K^+)^*$, and consequently

$$\begin{aligned}G^*G&=(C_\Phi-B_\Phi^*PA)^*(D_G^*D_G)^+(C_\Phi-B_\Phi^*PA)\\&=(C_\Phi-B_\Phi^*PA)^*(R_0-B_\Phi^*PB_\Phi)^+(C_\Phi-B_\Phi^*PA)\end{aligned}\tag{46}$$

$$\begin{aligned}KK^*&=(B_\Phi-AQC_\Phi^*)(D_KD_K^*)^+(B_\Phi-AQC_\Phi^*)^*\\&=(B_\Phi-AQC_\Phi^*)(R_0-C_\Phi QC_\Phi^*)^+(B_\Phi-AQC_\Phi^*)^*.\end{aligned}\tag{47}$$

The two Lyapunov equations in (42) and (43) now have the respective form of AREs

$$P=A^*PA+\Gamma_P^*(R_0-B_\Phi^*PB_\Phi)^+\Gamma_P \tag{48}$$
$$Q=AQA^*+\Gamma_Q(R_0-C_\Phi QC_\Phi^*)^+\Gamma_Q^* \tag{49}$$

where $\Gamma_P=(C_\Phi-B_\Phi^*PA)$ and $\Gamma_Q=(B_\Phi-AQC_\Phi^*)$. The following result is again translated from [1, Sec. IV, p. 495].

Lemma 4.2: Suppose that $\Phi(z)\geq 0$ for all $|z|=1$. Then all solutions P and Q to (48), and (49) respectively are nonnegative definite. There exist maximal solutions $P_{\max}$, $Q_{\max}$, and minimal solutions $P_{\min}$, $Q_{\min}$ to (48), and (49), respectively. All other solutions P, and Q to (48), and (49), respectively, satisfy $P_{\min}\leq P\leq P_{\max}$ and $Q_{\min}\leq Q\leq Q_{\max}$.

The solution sets corresponding to $P_{\min}$, and $Q_{\min}$ are associated with right, and left spectral factors of $\Phi(z)$, respectively, while $P_{\max}$, and $Q_{\max}$ are associated with those factors $W_G(z)$, and $W_K(z)$, whose transmission zeros are all outside unit circle, respectively. Any other solutions P and Q being neither minimal, nor maximal correspond to those factors $W_G(z)$ and $W_K(z)$ which contain some nonminimum phase zeros. The computation of $P_{\min}$ and $Q_{\min}$ is the main focus of this section, which yields the minimal degree spectral factors of $\Phi(z)$ in (39). We propose the following iterative algorithm.

- Set initial values $P_0=0$ and $Q_0=0$.
- For $k=0,1,\ldots,$ compute

$$P_{k+1}=A^*P_kA+\Gamma_{P_k}^*(R_0-B_\Phi^*P_kB_\Phi)^+\Gamma_{P_k} \tag{50}$$
$$Q_{k+1}=AQ_kA^*+\Gamma_{Q_k}(R_0-C_\Phi Q_kC_\Phi^*)^+\Gamma_{Q_k}^*. \tag{51}$$

- If $\|P_N-P_{N-1}\|$ is smaller than some pre-specified tolerance bound, terminate computation of $\{P_k\}$; If $\|Q_N-Q_{N-1}\|$ is smaller than the pre-specified tolerance bound, terminate computation of $\{Q_k\}$.

In the rest of the section we will show that the above algorithm is convergent with limit $P_{\min}$, and $Q_{\min}$. For this purpose define D_{G_m} and D_{K_m} as the minimum Cholesky factors via

$$\begin{aligned}D_{G_\mathrm{m}}^*D_{G_\mathrm{m}}&=R_0-B_\Phi^*P_{\min}B_\Phi\\D_{K_\mathrm{m}}D_{K_\mathrm{m}}^*&=R_0-C_\Phi Q_{\min}C_\Phi.\end{aligned}\tag{52}$$

Similarly define G_m and K_m as

$$\begin{aligned}G_\mathrm{m}&=(D_{G_\mathrm{m}}^+)^*(C_\Phi-B_\Phi^*P_{\min}A)\\K_\mathrm{m}&=(B_\Phi-AQ_{\min}C_\Phi^*)(D_{K_\mathrm{m}}^+)^*.\end{aligned}\tag{53}$$

Then, $(A,K_\mathrm{m},C_\Phi,D_{K_\mathrm{m}})$, and $(A,B_\Phi,G_\mathrm{m},D_{G_\mathrm{m}})$ are realizations associated with left, and right spectral factors of $\Phi(z)$, respectively. That is

$$W_{K_\mathrm{m}}(z)=\left[\begin{array}{c|c}A&K_\mathrm{m}\\\hline C_\Phi&D_{K_\mathrm{m}}\end{array}\right]$$
$$W_{G_\mathrm{m}}(z)=\left[\begin{array}{c|c}A&B_\Phi\\\hline G_\mathrm{m}&D_{G_\mathrm{m}}\end{array}\right]\tag{54}$$

are the left, and right spectral factors of $\Phi(z)$, respectively, and are thus outers. In light of Lemma 3.3 and Theorem 3.5, D_{G_m} has rank ρ, and by duality in Section III-B, D_{K_m} also has rank ρ. As a result, D_{G_m} and D_{K_m} have dimensions $\rho\times q$ and $q\times\rho$, respectively, and thus have the full rank. Recall that ρ is the normal rank of $\Phi(z)$. However for any other minimal degree factors $W_K(z)$ and $W_G(z)$ as in (41) which are not spectral factors of $\Phi(z)$, the associated D_G and D_K may have ranks strictly smaller than ρ. It is crucial to observe that the right spectral factor of $\Phi(z)$ can be obtained from the inner–outer factorization of $H(z)=W_G(z)$ as in Case i) of (6), and the left spectral factor of $\Phi(z)$ can be obtained from the inner–outer factorization of $H(z)=W_K(z)$ as in Case ii) of (6). Hence, the following result is true.

Theorem 4.3: Consider $W_G(z)$ of size $\rho\times q$, and $W_K(z)$ of size $q\times\rho$ as in (41), which are not spectral factors of $\Phi(z)$, but satisfy (40) with $\rho<q$, where $\Phi(z)\geq 0$ for all $|z|=1$. Denote

$$\begin{aligned}S_X&=B_\Phi^*XA+D_G^*G\quad S_Y=AY_kC_\Phi^*+KD_K^*\\\Pi_X&=D_G^*D_G+B_\Phi^*XB_\Phi\quad \Pi_Y=D_KD_K^*+C_\Phi Y_kC_\Phi^*.\end{aligned}$$

Then, for any $X^{(0)}\geq 0$ and $Y_0\geq 0$, the following DREs:

$$X^{(k+1)}=A^*X^{(k)}A+G^*G-S_{X^{(k)}}^*\Pi_{X^{(k)}}^+S_{X^{(k)}} \tag{55}$$
$$Y_{k+1}=AY_kA^*+KK^*-S_{Y_k}\Pi_{Y_k}^+S_{Y_k}^* \tag{56}$$

have solutions $\{X^{(k)}\}_{k=1}^T$, and $\{Y_k\}_{k=1}^T$, respectively, which are nonnegative definite. Suppose that $X^{(0)}\geq 0$ and $Y_0\geq 0$ are chosen such that $X^{(T)}$ converges to $X_{\max}\geq 0$, and Y_T converges $Y_{\max}\geq 0$, respectively, as $T\to\infty$, satisfying the AREs

$$X_{\max}=A^*X_{\max}A+G^*G-S_{X_{\max}}^*\Pi_{X_{\max}}^+S_{X_{\max}} \tag{57}$$
$$Y_{\max}=AY_{\max}A^*+KK^*-S_{Y_{\max}}\Pi_{Y_{\max}}^+S_{Y_{\max}}^*. \tag{58}$$

In this case, realizations of the left and right spectral factors in (54) are uniquely specified (up to a factor of unitary matrices), respectively, by

$$\begin{aligned}\Pi_{X_{\max}}&=D_{G_\mathrm{m}}^*D_{G_\mathrm{m}}=D_G^*D_G+B_\Phi^*X_{\max}B_\Phi\\G_\mathrm{m}&=(D_{G_\mathrm{m}}^+)^*(B_\Phi^*X_{\max}A+D_G^*G)\end{aligned}\tag{59}$$
$$\begin{aligned}\Pi_{Y_{\max}}&=D_{K_\mathrm{m}}D_{K_\mathrm{m}}^*=D_KD_K^*+C_\Phi Y_{\max}C_\Phi^*\\K_\mathrm{m}&=(AY_{\max}C_\Phi^*+KD_K^*)(D_{K_\mathrm{m}}^+)^*\end{aligned}\tag{60}$$

where D_{G_m} and D_{K_m} are the minimum rank Cholesky factors.

Proof: For factorizations in (40) with $W_K(z)$ and $W_G(z)$ given in (41), the inner–outer factorization of $H(z)=W_G(z)$ as in Case i) of (6) can be applied to obtain the right spectral factor $W_{G_\mathrm{m}}(z)=H_\mathrm{o}(z)$. Hence, the DRE in (55) is obtained using the iterative algorithm (13) with $D=D_G$, $C=G$, and $B=B_\Phi$, leading to the limit ARE in (57) as $T\to\infty$ under the hypothesis $X_k \to X_{\max}$. It follows that $D_{G_{\rm m}} = \Omega_{\rm m}$ and, thus, in light of Theorem 3.5

$$\begin{aligned} G_{\rm m} &= -D_{G_{\rm m}} F_{\rm m} = D_{G_{\rm m}} D_{G_{\rm m}}^{+} (D_{G_{\rm m}}^{+})^* S_{X_{\max}} \\ &= (D_{G_{\rm m}}^{+})^* S_{X_{\max}} = (D_{G_{\rm m}}^{+})^* (B_\Phi^* X_{\max} A + D_G^* G). \end{aligned}$$

Thus, (59) holds. A similar argument can be used to prove its dual in (60), which is skipped. □

Theorem 4.3 shows that the minimal solutions $P_{\min} \geq 0$, and $Q_{\min} \geq 0$ to the AREs (48) and (49) can be computed from

$$\begin{aligned} P_{\min} &= A^* P_{\min} A + G_{\rm m}^* G_{\rm m} \\ Q_{\min} &= A Q_{\min} A^* + K_{\rm m} K_{\rm m}^* \end{aligned} \quad (61)$$

respectively, which are basically the special cases of (42) and (43), respectively. It also indicates that

$$P_{\min} = A^* P_{\min} A + S_{X_{\max}}^* \Pi_{X_{\max}}^{+} S_{X_{\max}} \quad (62)$$

$$Q_{\min} = A Q_{\min} A^* + S_{Y_{\max}} \Pi_{Y_{\max}}^{+} S_{Y_{\max}}^* \quad (63)$$

in light of (59) and (60). Adding (62) to (57), and (63) to (58), respectively, yield

$$(X_{\max} + P_{\min}) = A^*(X_{\max} + P_{\min})A + G^* G \quad (64)$$

$$(Y_{\max} + Q_{\min}) = A(Y_{\max} + Q_{\min})A^* + K K^*. \quad (65)$$

Comparing the above two Lyapunov equations with those in (42) and (43), respectively, concludes that

$$P = X_{\max} + P_{\min} \quad Q = Y_{\max} + Q_{\min}. \quad (66)$$

Note that $X_{\max}$ is dependent on G, while $Y_{\max}$ is dependent on K, but $P_{\min}$ and $Q_{\min}$ are not. Hence we now switch to the notations

$$\begin{aligned} X_{\max} &= X_{\max}(G) \quad P = P(G) \\ Y_{\max} &= Y_{\max}(K) \quad Q = Q(K) \end{aligned}$$

respectively. The aforementioned analysis leads to

$$P(G) = X_{\max}(G) + P_{\min} \quad Q(K) = Y_{\max}(K) + Q_{\min} \quad (67)$$

which are associated with $W_G(z)$ and $W_K(z)$ in (41), respectively. It follows that

$$\begin{aligned} \Phi(z) &= D_G^* D_G + B_\Phi^* P(G) B_\Phi \\ &\quad + [D_G^* G + B_\Phi^* P(G) A](zI - A)^{-1} B_\Phi \\ &\quad + B_\Phi^* (z^{-1} I - A^*)^{-1} [D_G^* G + B_\Phi^* Q(G) A]^* \quad (68) \\ &= D_K D_K^* + C_\Phi Q(K) C_\Phi^* \\ &\quad + C_\Phi (zI - A)^{-1} [K D_K^* + A^* Q(K) C_\Phi^*] \\ &\quad + [G D_K^* + A^* Q(K) C_\Phi^*]^* (z^{-1} I - A^*)^{-1} C_\Phi^* \quad (69) \end{aligned}$$

where $P(G)$ and $Q(K)$ are given as in (67). We are now ready for the main result of this section.

Theorem 4.4: Let $\Phi(z)$ of size $q \times q$ as in (39) have normal rank $\rho < q$. Suppose that (A, B_Φ, C_Φ) is a minimal realization with A a stability matrix, and $\Phi(z) \geq 0$ and $\mathrm{rank}\{\Phi(z)\} = \rho$ for all $|z| = 1$. Then, the iterative formulas (50) and (51) in the proposed algorithm are convergent with limits $P_{\min}$, and $Q_{\min}$, which are the minimum solutions to the AREs (48) and (49), respectively.

Proof: We first prove that the limiting solution to (50) is $P_{\min}$. By (68) and $\Phi(z)$ in (39)

$$R_0 = D_G^* D_G + B_\Phi^* P(G) B_\Phi \quad C_\Phi = D_G^* G + B_\Phi^* P(G) A$$

for some G and D_G with $P(G) = P_{\min} + X_{\max}(G)$. Denote $\Delta_k = P(G) - P_k$. Substituting the above into (50) yields

$$P_{k+1} = A^* P_k A + S_{\Delta_k}^* \Pi_{\Delta_k}^{+} S_{\Delta_k}. \quad (70)$$

Because $P(G) = A^* P(G) A + G^* G$ by (42), the aforementioned equation leads to

$$\Delta_{k+1} = A^* \Delta_k A + G^* G - S_{\Delta_k}^* \Pi_{\Delta_k}^{+} S_{\Delta_k}$$

which is identical to (55) with $\Delta_{\rm i} = X^{(i)}$ for $i = k$ and $k+1$. Since $P_0 = 0$, $\Delta_0 = P(G) - P_0 \geq X \geq 0$ for any positive–semidefinite solution to ARE

$$X = A^* X A + G^* G - S_X^* \Pi_X^{+} S_X \quad (71)$$

which is the same ARE as in (57). In light of b) in Remark 3.2, and the results in the previous section, the iterative algorithm (13) is convergent with $X^{(k)} \to X_{\max}$ most likely. Moreover, the iterative algorithm in (50) is in fact independent of D_G and G. In other words for every possible pair of (D_G, G) the iterative solutions P_k in (50) satisfies

$$\Delta_k = P(G) - P_k = X^{(k)}$$

with $X^{(k)}$ the iterative solutions to (55). We may thus choose D_G and G such that $W_G(z)$ is strictly minimum phase, and D_G has full rank. Such D_G and G clearly exist by the hypothesis that $\Phi(z)$ has the rank r for all z on the unit circle, and by the minimality of (A, B_Φ, C_Φ). As such the positive semi-definite solution to the ARE in (71) is unique, which is $X_{\max}(G)$. As a result $\Delta_k = X^{(k)}$ is convergent, implying that $P_k = P(G) - \Delta_k$ is convergent to $P_{\min} = P(G) - X_{\max}(G)$ by (67). The proof for the limiting solution of (51) to $Q_{\min}$ can be shown similarly, which is omitted. □

It is noted that the convergence of the proposed spectral factorization algorithm embodied in (50) and (51) is established under the zero initial condition $P_0 = Q_0 = 0$. If $P_0 \geq 0$ and $Q_0 \geq 0$ are arbitrary, then the convergence of the DREs in (50) and (51) remains unknown, that is very different from the inner–outer factorization algorithms in the previous section.

Remark 4.5: In light of (67) and the proof of Theorem 4.4, we also obtain the right initial values $X^{(0)}$ and Y_0 for the iterative algorithms in (13), and (32), respectively, in order to ensure the limits $X_{\max}$ and $Y_{\max}$, respectively. That is, the initial condition $X^{(0)}$ as the solution to the Lyapunov equation (15) can ensure that the iterative algorithm in (13) admits limit $X_{\max}$, as required for the inner–outer factorization in Case i) of (6); Similarly if Y_0 satisfying $Y_0 = A Y_0 A^* + B B^*$ is chosen, then the iterative algorithm (32) admits the limit $Y_{\max}$, as required for the inner–outer factorization in Case ii) of (6). □

Remark 4.6: Regarding the generalized LQR control and Kalman filtering in Section III, stability of A can not be assumed in general, and thus Remark 4.5 does not apply. A simple way to bypass the instability issue is to set the control gain $F = F_1 + F_2$ where $(A + BF_1)$ is a stability matrix, and then consider the generalized LQR control for

$$\begin{aligned} x(t+1) &= (A + BF_1)x(t) + Bu(t) \\ z(t) &= (C + DF_1)x(t) + Du(t). \end{aligned} \tag{72}$$

This leads to the following modified algorithm:

$$\begin{aligned} F^{(k)} &= F_1 - (D^*D + B^*X^{(k)}B)^+ \\ &\quad \times [B^*X^{(k)}(A + BF_1) + D^*(C + DF_1)] \\ X^{(k+1)} &= (A + BF^{(k)})^*X^{(k)}(A + BF^{(k)}) \\ &\quad + (C + DF^{(k)})^*(C + DF^{(k)}) \end{aligned}$$

where $X^{(0)}$ satisfies the Lyapunov equation

$$X^{(0)} = (A + BF_1)^*X^{(0)}(A + BF_1) + (C + DF_1)^*(C + DF_1).$$

Remark 4.5 can be used to conclude that $X^{(k)} \to X_{\max}$ and $F^{(k)} \to F_{\mathrm{m}}$, as entailed for the generalized LQR control. Similarly, we can obtain the dual algorithm

$$\begin{aligned} L_k &= L_1 - [(A + L_1C)Y_kC^* + (B + L_1D)D^*] \\ &\quad \times (DD^* + CY_kC^*)^+ \\ Y_{k+1} &= (A + L_kC)Y_k(A + LC_k)^* \\ &\quad + (B + L_kD)(B + L_kD)^* \end{aligned}$$

where $(A + L_1C)$ is a stability matrix, and Y_0 satisfies the Lyapunov equation

$$Y_0 = (A + L_1C)Y_0(A + L_1C)^* + (B + L_kD)(B + L_kD)^*. \tag{73}$$

The convergence of Y_k to $Y_{\max}$, and L_k to L_{m} hold true as well. □

V. Illustrative Examples and Concluding Remarks

This paper considers generalized LQR control and Kalman filtering. The main contributions are the relations between these two optimization problems and computations of inner–outer factorizations (Section III), and spectral factorizations (Section IV). It is these relations that help develop iterative algorithms, convergent to the stabilizing solutions of the AREs, associated with generalized LQR control and Kalman filtering, which in turn solves the problem of inner–outer factorizations and spectral factorizations. In this section, we present two examples to demonstrate the proposed iterative algorithms, and their applications. Due to the space limit, we are unable to present the control example for tracking and sensitivity minimization. Interested readers are referred to [3].

Example 5.1: Our first example examines spectral factorization with the PSD modified from [19]

$$\Phi(z) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix} - vv^* + R_1z^{-1} + R_{-1}z \tag{74}$$

where $v^* = [\,0.1086 \quad 0.4052 \quad 0.9732\,]$ and

$$R_1 = R_{-1}^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}.$$

It can be verified that $\Phi(e^{j\omega})$ has rank 2 approximately with its third eigenvalue no greater than 10^{-5} for all ω. To apply the algorithm in (50), we set $A = 0$ of size 3×3 and

$$B_\Phi = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}.$$

It takes only two iterations for the algorithm to terminate with error tolerance 10^{-6} in computing $P_{\min}$. The right spectral factor is obtained via computing

$$D_{\mathrm{m}} = \begin{bmatrix} -0.395\,40 & 2.367\,95 & 0.000\,00 \\ -0.787\,74 & -0.131\,54 & 0.000\,03 \end{bmatrix}$$
$$G_{\mathrm{m}} = \begin{bmatrix} 0.410\,85 & -0.410\,85 & -0.205\,43 \\ -0.206\,22 & 0.206\,22 & 0.103\,11 \end{bmatrix}$$

as in (52) and (53), respectively. Since the spectral factors are unique up to a factor of unitary matrices, we have

$$UW_G(z) = \begin{bmatrix} 0.808\,47 & -0.000\,00 & -0.000\,00 \\ -0.351\,10 & 2.371\,60 & -0.000\,00 \end{bmatrix} + \begin{bmatrix} 0.183\,12 & -0.183\,12 & -0.091\,56 \\ 0.421\,66 & -0.421\,66 & -0.210\,83 \end{bmatrix} z^{-1}$$

is also a right spectral factor, which agrees with the example in [19], where

$$U = \begin{bmatrix} -0.055\,46 & -0.998\,46 \\ 0.998\,46 & -0.055\,46 \end{bmatrix}$$

is a unitary matrix. It is surprising to see the faster convergence in our proposed iterative algorithm. However, we need keep in mind that the highest power in this PSD matrix is only 1. Usually the number of iterations increases with respect to the highest power of $\Phi(z)$. □

We would like to comment that in practical numerical examples, the normal rank of the PSD matrix is almost always full, as in the above example. For this reason, we have to determine the normal rank numerically, which can be difficult in practice. On the other hand, it is possible that the normal rank is known in advance, and its inflation is due to the noise in estimation of the covariance data $\{R_k\}$, which is the case for blind channel estimation in [5], [13], and [15].

Example 5.2: This example is motivated from multiuser wireless data communications. Because of the multipath phenomena, the discretized wireless channels is represented by

$$H(z) = H_0 + H_1z^{-1} + \cdots + H_\ell z^{-\ell} \qquad H_k \in \mathbb{C}^{p \times m}. \tag{75}$$

Thus, the received signal data are convolution of the discretized channel with the transmitted digital data, plus the observation noise. As such it causes the problem of inter-symbol-interference (ISI) that poses the difficulty in symbol detection. A conventional approach to eliminate the ISI is through channel equalization [8], [9], [20]. Because redundancies are often introduced purposely in communications, the channel transfer matrix $H(z)$ is tall, i.e., $p > m$, and can be assumed to be strictly minimum phase. A simple state–space realization for $H(z)$ is

$$A = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \quad B = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$C = [H_1 \quad H_2 \quad \cdots \quad H_\ell] \quad D = H_0. \tag{76}$$

In [8] and [9], $D = H_0$ is assumed to have rank m and optimal channel equalizers are derived, which are the causal and stable left inverse of $H(z)$ that have the smallest $\mathcal{H}_2/\mathcal{H}_\infty$ norm. However due to the existence of the transmission delay, full-rank assumption on H_0 is not realistic. Hence, we apply the algorithm in Section III to compute the inner–outer factorization $H(z) = H_{\rm o}(z)H_{\rm i}(z)$, where $H_{\rm i}(z)$ is FIR, and has all its zeros at infinity, and $H_{\rm o}(\infty)$ has the full-column rank. Let d be the minimum integer such that $z^{-d}H_{\rm i}(z)^\sim$ is causal. Then, the optimal linear equalizer can be obtained as

$$G(z) = z^{-d}H_{\rm i}(z)^\sim H_{\rm o}^{(\rm inv)}(z) \quad \Longrightarrow \quad G(z)H(z) = z^{-d}I \tag{77}$$

where $H_{\rm o}^{(\rm inv)}(z)$ is the causal and stable left inverse of $H_{\rm o}(z)$, that has the smallest $\mathcal{H}_2$-norm.

In the following, we consider a numerical example with $H(z)$ of size 3×2, specified by $\ell = 3$ and

$$H_0 = \begin{bmatrix} 0.7233 & 1.0172 \\ 0.2104 & 0.2959 \\ -0.5664 & -0.7965 \end{bmatrix}$$
$$H_1 = \begin{bmatrix} -0.1199 & -0.5955 \\ -0.0653 & -0.1497 \\ 0.4853 & -0.4348 \end{bmatrix}$$
$$H_2 = \begin{bmatrix} -0.0793 & -1.3474 \\ 1.5352 & 0.4694 \\ -0.6065 & -0.9036 \end{bmatrix}$$
$$H_3 = \begin{bmatrix} 0.0359 & 0.5529 \\ -0.6275 & -0.2037 \\ 0.5354 & -2.0543 \end{bmatrix}.$$

Recall realization as in (76). Each element in D and C is generated as a normal random variable with zero mean, and unit variance, and the two columns of $D = H_0$ are linearly dependent. It can be verified that $H(z)$ is strictly minimum phase.

It is noted that in applying the iterative algorithm in (32), we do not need begin with initial condition $Y_0 = AY_0A^* + BB^*$, with (A, B, C, D) a realization of $H(z)$, due to the assumption that $H(z)$ is strictly minimum phase. In fact with $Y_0 = 0$, the iterative algorithm in (32) often converges faster, based on our numerical experience. For the given numerical example with the error tolerance 10^{-10}, the number of iterations is only 2, if $Y_0 = 0$; The number of iterations is 6, if $Y_0 = AY_0A^* + BB^*$. Both give the same $Y_{\max}$. With $Y_{\max}$ obtained, the inner and outer factors are computed according to Theorem 3.6. It turns out that the reachability gramian of $H_{\rm i}(z)$ has rank 1, implying that the minimal realization of $H_{\rm i}(z)$ has an order at most 1. Since $H(z)$ has only one zero outside the unit circle at infinity, $H_{\rm i}(z)$ has McMillan degree exactly 1. After eliminating the unreachable modes of $H_{\rm i}(z)$, we obtain

$$H_{\rm i}(z) = \begin{bmatrix} -0.5738 & -0.8070 \\ -0.0811 & -0.1140 \end{bmatrix} + z^{-1}\begin{bmatrix} -0.1399 \\ 0.9902 \end{bmatrix}[-0.8150 \quad 0.5795]$$

which is indeed an inner. The outer factor is specified by

$$\Omega_{\rm m} = \begin{bmatrix} -1.2013 & -0.4196 \\ -0.3548 & -0.0840 \\ 1.0584 & -0.5044 \end{bmatrix}$$
$$L_{\rm m} = \begin{bmatrix} -0.3341 & -0.0913 & 0.1324 \\ -0.4698 & -0.1284 & 0.1862 \\ -0.7230 & -0.1400 & -0.9752 \\ 0.5141 & 0.0995 & 0.6935 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The iterative algorithm works quite efficiently in terms of the computational complexity. □

As a final remark, we point out that the existence of the transmission zeros for $H(z)$ near the unit circle can lead to slow convergence in computation of the inner–outer factorizations, which can, in turn, cause the accumulation of the rounding-off errors in our proposed iterative algorithms. We suggest, in this case, to terminate the iteration before the rounding-off error grows large. One way to implement this is to employ the iterative algorithms in (13) and (50) simultaneously for inner–outer factorization in Case i) of (6) by setting

$$\left[\begin{array}{c|c} A & B_\Phi \\ \hline G & D_G \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and computing P, C_Φ, and R_0 in accordance with (42). The iterations need be terminated, once the difference between P and $(X^{(k)} + P_k)$ exceeds certain tolerance at the kth iteration. In light of (67) and the proof of Theorem 4.4, such a difference is zero for each k in absence of rounding-off errors, which is indeed observed in our numerical simulations for at least the first ten or more iterations. For the inner–outer factorization in Case ii) of (6), we may set

$$\left[\begin{array}{c|c} A & K \\ \hline C_\Phi & D_K \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and compute Q, B_Φ, and R_0 in accordance with (43). The iterations can be terminated, once the difference between Q and $(Y_{(k)} + Q_k)$ exceeds certain tolerance at the kth iteration. Because of the possible early termination, the precision of the numerical solution can be compromised due to the existence of the transmission zeros near the unit circle. The rounding-off error problem also exists for the spectral factorizations, but seems to be minor based on our simulation experience. One way to check the growing of the rounding-off error is to verify whether or not $(R_0 - B_\Phi^* P_k B_\Phi)$, or $(R_0 - C_\Phi Q_k C_\Phi^*)$ is sign positive. The iterative algorithm (50) needs be terminated, if $(R_0 - B_\Phi^* P_k B_\Phi)$ has negative eigenvalues, of which the absolute values exceed certain tolerance. Similarly the iterative algorithm (51) needs be terminated, if $(R_0 - C_\Phi Q_k C_\Phi^*)$ has negative eigenvalues, of which the absolute values exceed certain tolerance. Clearly, the ultimate solution to rounding-off errors is to speed up the convergence rate for our proposed iterative algorithms when the transmission zeros are near the unit circle, which is currently under study.

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