$$\leq -\sum_{j=1}^{r} \|z_j\|^2 + \sum_{j=1}^{r} s_j(\tilde{x}_1)\tilde{x}_1^2 + \frac{r}{4}\bar{c}_r^2 \tilde{L}_r^2 \tilde{x}_r^2 + \sum_{j=1}^{r} \tilde{x}_j^2 + b_r N(k_r)\dot{k}_r. \tag{5.8}$$

Therefore, choosing $\rho_i(\underline{x}_i, \underline{k}_{i-1}) \geq \tilde{L}_i^2(\underline{x}_i, \underline{k}_{i-1})$, $2 \leq i \leq r$, shows that, for $1 \leq i \leq r$, there exist some constants $c_i \geq 1$, such that

$$\begin{aligned} \dot{U}_i &\leq -\sum_{j=1}^{i} \|z_j\|^2 + b_i N(k_i)\dot{k}_i \\ &\quad + \left(\frac{i}{4}\bar{c}_i^2 + 1\right)\sum_{j=1}^{i} \rho_j \tilde{x}_j^2 + \bar{b}_i \tilde{x}_{i+1}^2 \\ &\leq -\sum_{j=1}^{i} \|z_j\|^2 + b_i N(k_i)\dot{k}_i + c_i \sum_{j=1}^{i+1} \dot{k}_j. \end{aligned} \tag{5.9}$$

We will now make use of the Lyapunov-like function U_i, $1 \leq i \leq r$, and inequality (5.9) to conclude the global boundedness of the state of the closed-loop system. For this purpose, we adapt a result from [11] which is stated here as a lemma for convenience.

Lemma 4.1: Let $U_i(\cdot)$ and $k_i(\cdot)$, $1 \leq i \leq r$, be smooth functions defined on $[0, t_f)$ with $U_i(t) \geq 0, \forall t \in [0, t_f)$, and $N(k_i) = \exp(k_i^2)\cos((\pi/2)k_i)$. For $t \in [0, t_f)$, if the following inequality holds:

$$U_i(t) \leq \int_0^t b_i(\tau) N(k_i(\tau))\dot{k}_i(\tau)d\tau + c_i \sum_{j=1}^{i+1} k_j(t) + \bar{c}_i \quad \forall t \in [0, t_f), 1 \leq i \leq r \tag{5.10}$$

where $b_i(t)$ is a time-varying parameter which takes values in the unknown closed intervals $I_i := [b_{m_i}, b_{M_i}]$ with $0 \notin I_i$, and c_i, $\bar{c}_i$, $1 \leq i \leq r$, represent some constants, then $U_i(t)$, $k_i(t)$, and $\int_0^t b_i(\tau)N(k_i(\tau))\dot{k}_i(\tau)d\tau$, $1 \leq i \leq r$, are bounded on $[0, t_f)$. ■

Now, similarly to the proof given in [11], assume the maximal interval of existence of the solution of the closed-loop system starting from any given initial condition is $[0, t_f)$ for some $t_f > 0$. Integrating inequality (5.9) and applying Lemma 4.1 shows that $U_i(t)$, $k_i(t)$, $\int_0^t b_i(\tau)N(k_i(\tau))\dot{k}_i(\tau)d\tau$, $1 \leq i \leq r$, are bounded on $[0, t_f)$. Thus, $V_i(z_i, t)$, and $\tilde{V}_i(\tilde{x}_i)$ are all bounded on $[0, t_f)$. Since $V_i(z_i, t)$ and $\tilde{V}_i(\tilde{x}_i)$ are proper positive–definite functions in z_i and $\tilde{x}_i$, $1 \leq i \leq r$, respectively, z_i, and $\tilde{x}_i$, $1 \leq i \leq r$, are also bounded on $[0, t_f)$. Therefore, finite-time escape cannot occur and $t_f = \infty$, that is, z_i and $\tilde{x}_i$, $1 \leq i \leq r$, are bounded for all $t \geq 0$.

Furthermore, α_i, as functions of $\tilde{x}_i$, and k_i, $1 \leq i \leq r$, are bounded for all $t \geq 0$, so are the variables x_i, $1 \leq i \leq r$. Therefore, $\dot{z}_i$, and $\dot{x}_i$, are bounded for all $t \geq 0$. Also, $\|\dot{\tilde{x}}\|$ is bounded for all $t \geq 0$. As a result, the inequality $\|\tilde{x}\|^2 = \sum_{i=1}^{r} \tilde{x}_i^2 \leq \sum_{i=1}^{r} \dot{k}_i$ implies that $\tilde{x}$ is square integrable on $[0, \infty)$. By Barbalat's lemma, $\tilde{x}$ approaches zero as $t \to \infty$. Thus, x approaches zero as $t \to \infty$.

Moreover, from (5.9), we can conclude that z_i, $1 \leq i \leq r$, are square integrable on $[0, \infty)$. This fact together with the boundedness of z_i, and $\dot{z}_i$, $1 \leq i \leq r$, implies that z_i, $1 \leq i \leq r$, approach zero as $t \to \infty$ by appealing to Barbalat's lemma.

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Relationship Between Perturbation Realization Factors With Queueing Models and Markov Models

Li Xia and Xi-Ren Cao

***Abstract*—Perturbation realization factor is an important concept in perturbation analysis of both queueing systems and Markov systems. A perturbation realization factor measures the effect of a perturbation on the system performance. This concept is important for the performance sensitivity and performance optimization of these systems. Since the perturbations in queueing systems are continuous in nature and those in Markov systems are discrete, it is not straightforward to establish the relationship between these two types of fundamental concepts. This note solves this long-standing problem. We find a formula that links these two types of perturbation realization factors in Gordon-Newell and open Jackson networks together. The results enhance our understanding of perturbation analysis and lead to new research directions.**

***Index Terms*—Markov decision processes (MDPs), performance potential, perturbation analysis (PA), perturbation realization factor, queueing systems.**

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I. Introduction

Perturbation analysis (PA) is an important approach to performance optimization of discrete event dynamic systems (DEDSs) [3], [7]. Perturbation realization factor measures the effect of a single perturbation on the system performance. It captures the flow of perturbations through DEDS and is as such an important concept in the PA theory. The performance sensitivities can be obtained by using the realization factors as building blocks.

PA was first proposed and extensively studied for queueing systems [1], [7] and later extended to Markov systems [2]. PA of queueing systems has clear intuitive interpretations and the algorithms for some problems are extremely efficient. PA of Markov systems applies to performance sensitivities with respect to all system parameters, and can be naturally extended to performance difference formulas that lead to policy iteration in the theory of Markov decision processes. Perturbation realization factors play an important role in both PA of queueing systems and Markov systems.

Since a queueing system (with exponentially distributed service times) can be viewed as a special Markov system [9], the realization factors for both systems must have some relationship. However, a perturbation in a queueing system, which refers to a small delay in time, is continuous in nature; and a perturbation in a Markov system, which refers to a change in system state, is discrete in nature. Therefore, it is not straightforward to establish the relationship between these two types of fundamental concepts.

This long-standing problem is solved in this note. We derive a formula that links the realization factors with queueing models and those with Markov models together. This study provides some new insights to the PA of both queueing-types of systems and Markov systems, which may lead to new ideas and research directions. With the relationship between both realization factors, we can easily establish parallel results for both types of systems. For example, we can establish the performance difference formulas for queueing systems. It is well known that the policy iteration algorithm follows directly from the performance difference formula. Therefore, it is natural that we may establish policy iteration-based optimization methods for queueing systems. The results of the PA of Markov systems enable us to develop sensitivity formulas and optimization approaches for queueing systems.

The rest of the note is organized as follows. In Section II, we review the fundamentals for realization factors. In Section III, we derive the relationship formula between the two types of realization factors for Gordon–Newell networks, first intuitively from the conceptual meanings of realization factors, and then mathematically from the existing sensitivity formulas. In Section IV, we further extend the results to open Jackson networks. In Section V, we conclude the note with some discussions.

II. Background on Perturbation Realization Factors

We first review the concept of realization factors in PA of queueing systems and the related performance derivative equations. We then review the concepts of performance potential, realization factors, and Poisson equation in the theory of Markov systems.

A. Realization Factors in PA of Queueing Systems

We consider a Gordon–Newell network (it is also called a closed Jackson network) [4], [6] consisting of M servers with exponentially distributed service times with service rates μ_i, $i = 1, 2, \ldots, M$. The total number of customers in the network is N. After the service completion at server i, a customer will leave server i and enter the buffer of server j with probability q_{ij}, $j = 1, 2, \ldots, M$. Without loss of generality, we suppose $q_{ii} = 0$. The system state is denoted as $\mathbf{n} = (n_1, n_2, \ldots, n_i, \ldots, n_M)$, where n_i is the number of customers at server i. The state space is $\mathcal{S} = \{all\ \mathbf{n} : \sum_{i=1}^{M} n_i = N\}$. Let $\mathbf{n}(t)$ denote the system state at time t, $f(\mathbf{n})$ be the cost function which maps the state space $\mathcal{S}$ to $\mathcal{R} = (-\infty, \infty)$, T_L be the Lth service completion time of the network, and η be the *time-average performance* defined as

$$\eta = \lim_{L\to\infty} \frac{\int_0^{T_L} f(\mathbf{n}(t))\, dt}{T_L} = \lim_{L\to\infty} \frac{F_L}{T_L} \tag{1}$$

with $F_L := \int_0^{T_L} f(\mathbf{n}(t))dt$. We use $\eta^{(f)}$ to denote the *customer-average performance*

$$\eta^{(f)} = \lim_{L\to\infty} \frac{F_L}{L}. \tag{2}$$

We assume that the state process $\mathbf{n}(t)$ is ergodic so that all the limits exist.

We study the effect of a single perturbation of a service completion time on the system performance. As we know, when a perturbation of a server is propagated in an irreducible closed network, it will either be realized or lost with probability one [1]. The probability that it is realized is called the *perturbation realization probability*. It is dependent on the system state when the perturbation is generated and the server which is perturbed. We denote the realization probability of a perturbation at server k when the system is at state $\mathbf{n}$ as $c(\mathbf{n}, k)$, $\mathbf{n} \in \mathcal{S}$, $k = 1, 2, \ldots, M$.

When server k is perturbed, the system performance $\eta^{(f)}$ will also be affected. In PA theory, we define the realization factor of a perturbation Δ of server k at state $\mathbf{n}$ for $\eta^{(f)}$ as

$$\begin{aligned} c^{(f)}(\mathbf{n}, k) &= \lim_{L\to\infty} \lim_{\Delta\to 0} E\left\{\frac{\Delta F_L}{\Delta}\right\} \\ &= \lim_{L\to\infty} \lim_{\Delta\to 0} E\left\{\frac{F_L' - F_L}{\Delta}\right\} \\ &= \lim_{L\to\infty} \lim_{\Delta\to 0} E \\ &\quad \left\{\frac{1}{\Delta}\left(\int_0^{T_L'} f\left(\mathbf{n}'(t)\right)dt - \int_0^{T_L} f(\mathbf{n}(t))dt\right)\right\} \end{aligned} \tag{3}$$

where $\mathbf{n}'(t)$ is the perturbed system state at time t, T_L' is the time when the perturbed system has served L customers. From (3), the perturbation realization probability $c(\mathbf{n}, k)$ can be viewed as a special $c^{(f)}(\mathbf{n}, k)$ with $f(\mathbf{n}) = I(\mathbf{n}) \equiv 1$ for all $\mathbf{n} \in \mathcal{S}$.

With realization factors $c^{(f)}(\mathbf{n}, k)$, we can derive the system performance sensitivity with respect to system parameters. We have [1]

$$\frac{d\eta^{(f)}}{d\mu_k} = -\frac{\eta^{(I)}}{\mu_k} \sum_{\mathbf{n}\in\mathcal{S}} \pi(\mathbf{n}) c^{(f)}(\mathbf{n}, k) \tag{4}$$

where $\eta^{(I)}$ is a special system performance corresponding to $f(\mathbf{n}) = I(\mathbf{n}) \equiv 1$ for all $\mathbf{n} \in \mathcal{S}$, and $\pi(\mathbf{n})$ is the steady-state probability of state $\mathbf{n}$. The derivative of $\eta^{(I)}$ is

$$\frac{d\eta^{(I)}}{d\mu_k} = -\frac{\eta^{(I)}}{\mu_k} \sum_{\mathbf{n}\in\mathcal{S}} \pi(\mathbf{n}) c(\mathbf{n}, k). \tag{5}$$

Derivative formulas (4) and (5) are the basis of the gradient-based performance optimization schemes of queueing systems [1], [5].

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B. Realization Factors in Markov Systems

The PA theory has been extended to Markov systems in the past decade. Now, we review the related results for Markov processes [2].

We denote an irreducible Markov process as $\mathrm{X} = \{X_t, t \geq 0\}$, where X_t is the system state at time t. The finite state space is $\mathcal{S} = \{1, 2, \ldots, S\}$ and the infinitesimal generator is $B = [b(u,v)]_{S\times S}$, where $b(u,v) \geq 0$ if $u \neq v$, $b(u,u) = -\sum_{v\neq u} b(u,v)$, $u, v \in \mathcal{S}$. Let π denote the row vector of the steady-state probability, and e be an S-dimensional column vector whose elements are all one. We have $Be = 0$, $\pi e = 1$, and $\pi B = 0$. Let f be a column vector of the performance function, and $\eta = \pi f$ be the time-average performance.

The *performance potential* g measures the contribution of the initial states to the system performance. g is a column vector and its element $g(u)$, $u \in \mathcal{S}$, is defined as

$$g(u) = \lim_{T\to\infty} E\left\{ \int_0^T [f(X_t) - \eta]\, dt | X_0 = u \right\}. \tag{6}$$

From (6), we can derive the Poisson equation [2]

$$Bg = -f + \eta e. \tag{7}$$

The *perturbation realization factor* is defined as the difference of the performance potentials between any two states

$$\begin{aligned} d(u,v) &= g(v) - g(u) \\ &= \lim_{T\to\infty} E\left\{ \int_0^T \left[f\left(X'_t\right) - f(X_t)\right] dt | X_0 = u, X'_0 = v \right\}. \end{aligned} \tag{8}$$

$d(u,v)$ reflects the difference of the contributions of two initial states to the total system performance, i.e., it measures the effect on the system performance if the system is perturbed from state u to state v.

With the Poisson equation (7), we can easily derive the performance difference equation. Let B and B' be two ergodic infinitesimal generators on $\mathcal{S}$, g be the performance potential of B, π' be the steady-state probability of B', and $\eta' = \pi' f$ be the performance of B'. Set $\Delta B = B' - B$. Then

$$\eta' - \eta = \pi' \Delta B g. \tag{9}$$

Next, we set $B(\delta) = (1-\delta)B + \delta B' = B + \delta \Delta B$. Let $\eta(\delta) = \pi(\delta) f$ be the performance of the Markov process with infinitesimal generator $B(\delta)$. Then, the system performance derivative with respect to δ is

$$\frac{d\eta}{d\delta} = \pi \Delta B g. \tag{10}$$

Equations (9) and (10) are the fundamental formulas for performance optimization of Markov systems. The details can be found in [2].

III. Relationship Between Two Realization Factors

Since a queueing system can be viewed as a special Markov system, we expect that $d(u,v)$ and $c^{(f)}(\mathbf{n}, k)$ have some relationship. We study this relationship in this section.

We first establish the relationship by using the meanings of realization factors in both systems. This relatively "intuitive" analysis helps to derive and explain the results but is not rigorous. The mathematical proof will be provided later.

From the definition (3) we have (with $\Delta > 0$, we have $T'_L \geq T_L$)

$$\begin{aligned} c^{(f)}(\mathbf{n}, k) &= \lim_{L\to\infty} \lim_{\Delta\to 0} E \\ &\quad \left\{ \frac{1}{\Delta} \left(\int_0^{T'_L} f\left(\mathbf{n}'(t)\right) dt - \int_0^{T_L} f\left(\mathbf{n}(t)\right) dt \right) \right\} \\ &= \lim_{L\to\infty} \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{ \int_0^{T_L} \left[f\left(\mathbf{n}'(t)\right) - f\left(\mathbf{n}(t)\right)\right] dt \right\} \\ &\quad + \lim_{L\to\infty} \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{ \int_{T_L}^{T'_L} f\left(\mathbf{n}'(t)\right) dt \right\} \\ &= \lim_{T\to\infty} \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{ \int_0^{T} \left[f\left(\mathbf{n}'(t)\right) - f\left(\mathbf{n}(t)\right)\right] dt \right\} \\ &\quad + \lim_{L\to\infty} \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{ \int_{T_L}^{T'_L} f\left(\mathbf{n}'(t)\right) dt \right\}. \end{aligned} \tag{11}$$

For a rough analysis, let us assume that we can freely change the order of the two limits, the expectation and the integration in each term of (11). We first consider the second term on the right-hand side of (11). As $L \to \infty$, the system will reach its steady state and we have $\lim_{t\to\infty} E\{f(\mathbf{n}'(t))\} = \lim_{t\to\infty} E\{f(\mathbf{n}(t))\} = \eta$. So as L becomes large, we may roughly have $E\{\int_{T_L}^{T'_L} f(\mathbf{n}'(t))dt\} \approx \eta E(T'_L - T_L)$. Therefore

$$\lim_{L\to\infty} \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{ \int_{T_L}^{T'_L} f\left(\mathbf{n}'(t)\right) dt \right\} = \lim_{L\to\infty} \lim_{\Delta\to 0} \frac{1}{\Delta} \eta E\left\{ T'_L - T_L \right\}. \tag{12}$$

By the definition of the perturbation realization probability, we have

$$c(\mathbf{n}, k) = \lim_{L\to\infty} \lim_{\Delta\to 0} \frac{E\left\{T'_L - T_L\right\}}{\Delta}. \tag{13}$$

Therefore

$$\lim_{L\to\infty} \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{ \int_{T_L}^{T'_L} f\left(\mathbf{n}'(t)\right) dt \right\} = c(\mathbf{n}, k)\eta. \tag{14}$$

Now, let us focus on the first term of the right-hand side of (11). Roughly speaking, by the concept of the realization factor $d(u,v)$ for the Markov systems (see (8), with u, v understood as the states of the queueing system), the difference $\lim_{T\to\infty} E\{\int_\Delta^T [f(\mathbf{n}'(t)) - f(\mathbf{n}(t))]dt\}$ can be determined by the realization factors $d(\mathbf{n}(\Delta), \mathbf{n}'(\Delta))$, with $\Delta \to 0$. Therefore, we need to determine the states $\mathbf{n}(\Delta)$ and $\mathbf{n}'(\Delta)$.

By assumption, at time $t = 0$, the system is at state $\mathbf{n}$ and server k obtains a small perturbation Δ. This can be viewed as that server k is "frozen" (no service provided and hence no service completion) in the time period $[0, \Delta)$ on the perturbed sample path $\mathbf{n}'(t)$. Therefore, the difference between $\mathbf{n}(t)$ and $\mathbf{n}'(t)$ is that on $\mathbf{n}(t)$ all servers may complete service in $[0, \Delta)$, but on $\mathbf{n}'(t)$ all servers except server k may complete service in $[0, \Delta)$. We do not consider the possibility that more than one server complete service in $[0, \Delta)$ since its probability is at the order of Δ^2 (the number of servers is finite). Thus, at time $t = \Delta$, $\mathbf{n}(t)$ is

$$\begin{array}{ll} at\ state\ \mathbf{n}_{ij}, & \text{with probability} \\ \quad \epsilon(n_i)\mu_i q_{ij}\Delta, & i, j = 1, 2, \ldots, M \quad (15) \\ \text{at state } \mathbf{n}, & \text{with probability} \\ \quad \left\{1 - \sum_{i=1}^{M} \epsilon(n_i)\mu_i \Delta\right\} & \quad (16) \end{array}$$

where $\mathbf{n}_{ij} = (n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_M)$ is a neighboring state of $\mathbf{n}$ with $n_i \geq 1$, $\epsilon(n_i)$ is an indicator function which is defined as $\epsilon(n_i) = 0$ (or 1) if $n_i = 0$ (or > 0). Similarly, at $t = \Delta$, $\mathbf{n}'(t)$ is

$$\begin{array}{ll} \text{at state } \mathbf{n}_{ij}, & \text{with probability} \\ \quad \epsilon(n_i)\mu_i q_{ij}\Delta, & i, j = 1, 2, \ldots, M,\ i \neq k \quad (17) \\ \text{at state } \mathbf{n}, & \text{with probability} \\ \quad \left\{1 - \sum_{i \neq k}^{M} \epsilon(n_i)\mu_i \Delta\right\}. & \quad (18) \end{array}$$

Therefore, at time $t = \Delta$, ignoring the high-order terms of Δ, we conclude that with probability $\epsilon(n_i)\mu_i q_{ij}\Delta$, $\mathbf{n}(t)$ is at $\mathbf{n}_{ij}$ and $\mathbf{n}'(t)$ is at $\mathbf{n}$, $i, j = 1, 2, \ldots, M$; with probability $\epsilon(n_i)\mu_i q_{ij}\Delta$, $\mathbf{n}(t)$ is at $\mathbf{n}$ and $\mathbf{n}'(t)$ is at $\mathbf{n}_{ij}$, $i, j = 1, 2, \ldots, M$, $i \neq k$; and with probability $1 - \sum_{i=1}^{M} \epsilon(n_i)\mu_i\Delta - \sum_{i \neq k}^{M} \epsilon(n_i)\mu_i\Delta$, both $\mathbf{n}(t)$ and $\mathbf{n}'(t)$ are at $\mathbf{n}$. From the meaning of realization factors in (8), we should have

$$\begin{aligned} &\lim_{T\to\infty} E\left\{\int_{\Delta}^{T} \left[f\left(\mathbf{n}'(t)\right) - f\left(\mathbf{n}(t)\right)\right] dt\right\} \\ &= \sum_{i,j=1}^{M} \left\{\epsilon(n_i)\mu_i q_{ij}\Delta\right\} d(\mathbf{n}_{ij}, \mathbf{n}) \\ &\quad + \sum_{i,j=1, i\neq k}^{M} \left\{\epsilon(n_i)\mu_i q_{ij}\Delta\right\} d(\mathbf{n}, \mathbf{n}_{ij}) \\ &= \epsilon(n_k)\mu_k\Delta \sum_{j=1}^{M} q_{kj} d(\mathbf{n}_{kj}, \mathbf{n}). \end{aligned} \tag{19}$$

From (11), (14), and (19), we get

$$\begin{aligned} c^{(f)}(\mathbf{n}, k) - c(\mathbf{n}, k)\eta &= \epsilon(n_k)\mu_k \sum_{j=1}^{M} q_{kj} d(\mathbf{n}_{kj}, \mathbf{n}) \\ &\quad + \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{\int_{0}^{\Delta} \left[f\left(\mathbf{n}'(t)\right) - f\left(\mathbf{n}(t)\right)\right] dt\right\}. \end{aligned} \tag{20}$$

We can use the same argument as in (19) to evaluate the last term in (20). For any $0 \leq t \leq \Delta$, we have

$$\begin{aligned} &E\left\{f\left(\mathbf{n}'(t)\right) - f\left(\mathbf{n}(t)\right)\right\} \\ &= \sum_{i,j=1}^{M} \left\{\epsilon(n_i)\mu_i q_{ij} t\right\} \left[f(\mathbf{n}) - f(\mathbf{n}_{ij})\right] \\ &\quad + \sum_{i,j=1, i\neq k}^{M} \left\{\epsilon(n_i)\mu_i q_{ij} t\right\} \left[f(\mathbf{n}_{ij}) - f(\mathbf{n})\right] \\ &= \epsilon(n_k)\mu_k t \sum_{j=1}^{M} q_{kj} \left[f(\mathbf{n}) - f(\mathbf{n}_{kj})\right]. \end{aligned} \tag{21}$$

From (21), we have $|E\{f(\mathbf{n}'(t)) - f(\mathbf{n}(t))\}| < 2G\mu_k\Delta$, where G is the upper bound of $|f(\mathbf{n})|$, i.e., $|f(\mathbf{n})| < G$, $\mathbf{n} \in \mathcal{S}$. Then, it can be easily verified that the last term in (20) is zero and we have

$$\boxed{c^{(f)}(\mathbf{n}, k) - c(\mathbf{n}, k)\eta = \epsilon(n_k)\mu_k \sum_{j=1}^{M} q_{kj} d(\mathbf{n}_{kj}, \mathbf{n}).} \tag{22}$$

This formula describes the relationship between realization factors defined with the queueing model and those with the Markov model. The meaning of this formula can be understood from the above analysis procedure. Generally speaking, both sides of this formula describe the average effect of a perturbation at server k. The left-hand side of the formula quantifies the perturbation effect from the traditional PA theory in queueing systems. The right-hand side of the formula quantifies the effect from the perturbation due to state jumps at server k in a time unit. This formula bridges the gap between PA of queueing systems and Markov potential theory.

The previous derivation is very rough especially we have exchanged the order of limits, expectations and integrations at our wish. Now, we rigorously prove this relationship formula with the existing mathematical formulas. The performance derivatives of Gordon–Newell networks can be derived in two ways: from the perturbation analysis with realization factors $c^{(f)}(\mathbf{n}, k)$, and from the Markov theory with potential-based realization factors $d(\mathbf{n}, \mathbf{n}')$. With these two approaches, we can formally prove the relationship formula (22).

We consider a more general closed network which is the same as the one discussed in Section II-A except that the service rate of server k may depend on the system state. When the system state is $\mathbf{n}$ the service rate of server k is denoted as $\mu_{k,\mathbf{n}}$, $\mathbf{n} \in \mathcal{S}$. Now, we suppose that the service rate of server k changes from $\mu_{k,\mathbf{n}}$ to $\mu_{k,\mathbf{n}} + \Delta\mu_{k,\mathbf{n}}$. It is known that the derivative of the customer-average performance $\eta^{(f)}$ is [1]

$$\frac{d\eta^{(f)}}{d\mu_{k,\mathbf{n}}} = -\frac{\eta^{(I)}}{\mu_{k,\mathbf{n}}} \pi(\mathbf{n}) c^{(f)}(\mathbf{n}, k). \tag{23}$$

Similarly, the derivative of $\eta^{(I)}$ with respect to the state-dependent service rate $\mu_{k,\mathbf{n}}$ is

$$\frac{d\eta^{(I)}}{d\mu_{k,\mathbf{n}}} = -\frac{\eta^{(I)}}{\mu_{k,\mathbf{n}}} \pi(\mathbf{n}) c(\mathbf{n}, k). \tag{24}$$

Since $\eta^{(f)} = \lim_{L\to\infty}(F_L/L) = \lim_{L\to\infty}(F_L/T_L)(T_L/L) = \eta\eta^{(I)}$, we have

$$\frac{d\eta^{(f)}}{d\mu_{k,\mathbf{n}}} = \frac{d\left(\eta\eta^{(I)}\right)}{d\mu_{k,\mathbf{n}}} = \frac{d\eta}{d\mu_{k,\mathbf{n}}}\eta^{(I)} + \frac{d\eta^{(I)}}{d\mu_{k,\mathbf{n}}}\eta. \tag{25}$$

So the derivative of the time-average performance η with respect to $\mu_{k,\mathbf{n}}$ is

$$\begin{aligned} \frac{d\eta}{d\mu_{k,\mathbf{n}}} &= \frac{1}{\eta^{(I)}} \left[\frac{d\eta^{(f)}}{d\mu_{k,\mathbf{n}}} - \frac{d\eta^{(I)}}{d\mu_{k,\mathbf{n}}}\eta\right] \\ &= -\frac{\pi(\mathbf{n})}{\mu_{k,\mathbf{n}}} \left[c^{(f)}(\mathbf{n}, k) - c(\mathbf{n}, k)\eta\right]. \end{aligned} \tag{26}$$

Now, let us use the Markov potential theory to derive the derivative of η with respect to $\mu_{k,\mathbf{n}}$. The state process can be viewed as a Markov process with infinitesimal generator B, which can be determined by $\mu_{i,\mathbf{n}}$ and q_{ij}, $i, j = 1, 2, \ldots, M$ and $\mathbf{n} \in \mathcal{S}$. Let $g(\mathbf{n})$,

$\mathbf{n} \in \mathcal{S}$, be the performance potential. Now, we pick up a particular state denoted as $\mathbf{n}$. When the service rate of server k changes from $\mu_{k,\mathbf{n}}$ to $\mu_{k,\mathbf{n}} + \Delta\mu_{k,\mathbf{n}}$, the infinitesimal generator changes from B to $B(\Delta\mu_{k,\mathbf{n}}) = B + \Delta\mu_{k,\mathbf{n}}\Delta B$, where the high order of $\Delta\mu_{k,\mathbf{n}}$ is neglected. Comparing with (10), $\Delta\mu_{k,\mathbf{n}}$ can be viewed as δ in (10).

From the structure of the infinitesimal generator in Gordon–Newell networks, $\mu_{k,\mathbf{n}}$ only determines the transition probabilities from state $\mathbf{n}$. Thus, when $\mu_{k,\mathbf{n}}$ changes, it only affects one row in the transition probability matrix. That is, all the elements of $\Delta\mu_{k,\mathbf{n}}\Delta B$ are zero except the row corresponding to state $\mathbf{n}$, denoted as $\Delta\mu_{k,\mathbf{n}}\Delta B(\mathbf{n},*)$. For this row, the value of the diagonal element is $\Delta\mu_{k,\mathbf{n}}\Delta B(\mathbf{n},\mathbf{n}) = -\Delta\mu_{k,\mathbf{n}}\epsilon(n_k)$; the value for the neighboring states of $\mathbf{n}$ is $\Delta\mu_{k,\mathbf{n}}\Delta B(\mathbf{n},\mathbf{n}_{kj}) = q_{kj}\Delta\mu_{k,\mathbf{n}}\epsilon(n_k)$, $j = 1, 2, \ldots, M$; and all the other elements are zero. Thus, we get the value of the matrix ΔB. From (10), we can derive the performance derivative as

$$\begin{aligned}\frac{d\eta}{d\mu_{k,\mathbf{n}}} &= \pi\Delta B g \\ &= \epsilon(n_k)\pi(\mathbf{n})\left[\sum_{j=1}^{M} q_{kj}g(\mathbf{n}_{kj}) - g(\mathbf{n})\right] \\ &= \epsilon(n_k)\pi(\mathbf{n})\sum_{j=1}^{M} q_{kj}\left[g(\mathbf{n}_{kj}) - g(\mathbf{n})\right] \\ &= \epsilon(n_k)\pi(\mathbf{n})\sum_{j=1}^{M} q_{kj}d(\mathbf{n},\mathbf{n}_{kj}).\end{aligned} \tag{27}$$

Comparing (26) and (27), we obtain the relationship formula

$$c^{(f)}(\mathbf{n},k) - c(\mathbf{n},k)\eta = \epsilon(n_k)\mu_{k,\mathbf{n}}\sum_{j=1}^{M} q_{kj}d(\mathbf{n}_{kj},\mathbf{n}). \tag{28}$$

In fact, (22) can be viewed as a special case of (28). If the service rates are state independent, i.e., $\mu_{k,\mathbf{n}} = \mu_k$ for all $\mathbf{n} \in \mathcal{S}$, (28) becomes (22).

IV. Extension to Open Jackson Networks

We have established the relationship between the two types of realization factors defined with two different models for Gordon–Newell networks. In this section, we will show that these results hold for open Jackson networks after some minor notational modifications.

Compared with Gordon–Newell networks, the main difference in open Jackson networks is that there is a customer arrival process from the outside to the network. The customer arrival process is assumed to be a Poisson process with rate μ_0. When a customer arrives, it will enter the buffer of server i with probability q_{0i}, $i = 1, 2, \ldots, M$. After its service completion at server i, a customer will depart from the network with probability q_{i0} and enter server j with probability q_{ij}, $j = 1, 2, \ldots, M$ [8]. Since the number of states is infinite, we require the cost function f be bounded.

We view the customer arrival source as server 0. As we know, server 0 can be considered as a server containing infinitely many customers. In this sense, an open Jackson network consisting of M servers can be viewed as a Gordon–Newell network consisting of $M+1$ servers with server 0 having infinitely many customers [1].

It is easy to check that all the previous analyses and results preserve correctness. The only thing we need to change is that the servers should be counted from 0 to M. The relationship formula (22) becomes

$$c^{(f)}(\mathbf{n},k) - c(\mathbf{n},k)\eta = \epsilon(n_k)\mu_k\sum_{j=0}^{M} q_{kj}d(\mathbf{n}_{kj},\mathbf{n}) \tag{29}$$

where we define $\epsilon(n_0) = 1$, $q_{00} = 0$, $\mathbf{n}_{k0} = (n_1, \ldots, n_k - 1, \ldots, n_M)$, $\mathbf{n}_{0k} = (n_1, \ldots, n_k + 1, \ldots, n_M)$, and $k = 0, 1, \ldots, M$.

Furthermore, it is known that the realization probability of open Jackson networks has the special property [1]

$$\begin{aligned}c(\mathbf{n},0) &= 1 \\ c(\mathbf{n},k) &= 0, \qquad k \neq 0.\end{aligned} \tag{30}$$

This formula can be understood from the analysis of perturbation propagation in open Jackson networks. With (30), the relationship formula will have a simpler format

$$\begin{aligned}c^{(f)}(\mathbf{n},0) - \eta &= \mu_0\sum_{j=1}^{M} q_{0j}d(\mathbf{n}_{0j},\mathbf{n}) \\ c^{(f)}(\mathbf{n},k) &= \epsilon(n_k)\mu_k\sum_{j=0}^{M} q_{kj}d(\mathbf{n}_{kj},\mathbf{n}), \\ &\qquad k = 1, 2, \ldots, M.\end{aligned} \tag{31}$$

V. Conclusion

This note solves a long-standing problem in perturbation analysis: It establishes the relationship between the realization factors defined with the queueing model, which measure the effect of continuous perturbations, and the realization factors defined with the Markov model, which measure the effect of discrete perturbations. This study enhances our understanding of perturbation analysis of both queueing-types of systems and Markov systems. The results may lead to new ideas and new research directions. One direct consequence is, the study builds up a bridge between perturbation analysis of both systems and hance allows us to establish parallel results for both types of systems. For example, we can establish the performance difference formulas for queueing systems. It is well known that the policy iteration algorithm follows directly from the performance difference formula. Therefore, it is natural that we may establish policy iteration-based optimization methods for queueing systems. This approach is new to the optimization of queueing systems [10]. Also, since the realization factors $d(u,v)$ for Markov systems can be applied to the performance sensitivities with respect to all the system parameters, with the relationship formulas, we may be able to use the realization factors $c^{(f)}(\mathbf{n},k)$ for queueing systems to establish performance sensitivity formulas for parameters other than mean service times, such as routing probabilities, which is unsolved by the current PA approach for queueing systems. These are new directions that require further investigations.

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A Method for Nonlinear Least Squares With Structured Residuals

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***Abstract*—This note develops a modification of standard nonlinear least squares methods with reduced sensitivity to the quality of the initial guess. The technique is presented in the context of least squares fitting of dynamic system models, but may apply to other kinds of problems. The performance of the technique is compared to standard methods for a variety of test problems.**

***Index Terms*—Nonlinear least squares, optimization, system identification.**

I. INTRODUCTION

Nonlinear least squares methods are often used to fit dynamic system models to experimental data. Conventional methods applied to these problems tend to be susceptible to local minima, have unpredictable performance for initial guesses that are far from the desired minimum, and do not take advantage of the structure of residuals in data fitting problems. The method developed in this note uses residual structure and offers convergence from a wider range of initial guesses to the desired minimum than standard techniques such as Levenburg–Marquardt or Gauss–Newton. In addition, the performance of the proposed method is easy to predict in some cases.

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The unconstrained nonlinear least squares estimate can be expressed as the argument of the minimization over parameter vectors μ

$$\hat{\mu} = \arg\min_{\mu} V(\mu) \tag{1}$$

where $V(\mu)$ is the "sum-of-squared errors" loss function

$$V(\mu) = \frac{1}{2} r^T r = \frac{1}{2} \sum_{k=1}^{N} r_k^2. \tag{2}$$

For the common output error data-fitting application, the residual vector r is the difference between a model response $\hat{y}$, which depends on the parameters μ and inputs, and a set of observations y. We assume that successive values r_k in r represent errors at increasing time values. A requirement for the minimum in (1) is that the gradient of $V(\mu)$ be zero. The gradient can be written

$$g(\mu) = \sum_{k=1}^{N} \begin{pmatrix} \frac{\partial r_k}{\partial \mu_1} & \frac{\partial r_k}{\partial \mu_2} & \cdots & \frac{\partial r_k}{\partial \mu_M} \end{pmatrix}^T r_k \tag{3}$$

$$= J^T r \tag{4}$$

where M is the number of parameters and J is the Jacobian matrix of the residual with respect to the parameters. Newton's method can applied to (3) to find a series of iterates $\mu^{(i)}$ that can be evaluated by computer to solve for $g(\mu^{(i)}) = 0$, i.e.,

$$\mu^{(i+1)} = \mu^{(i)} - \left[\nabla g\left(\mu^{(i)}\right)\right]^{-1} g\left(\mu^{(i)}\right). \tag{5}$$

This is the "full-Newton" update for solving the nonlinear least-squares problem [1]. The difficulty in this equation is the Hessian matrix $\nabla g(\mu^{(i)})$, which can be written

$$\nabla g\left(\mu^{(i)}\right) = J^T J + \text{second-order derivative terms} \tag{6}$$

where the second order derivative terms are often inconvenient to compute. The popular Levenberg–Marquardt method uses (5), except that the second-order terms are replaced with a continuation parameter λ and weight matrix D, i.e.,

$$\mu^{(i+1)} = \mu^{(i)} - \left[J^T J + \lambda D\right]^{-1} J^T r \tag{7}$$

where J and r are evaluated at $\mu^{(i)}$. The Gauss–Newton method uses (7) with $\lambda = 0$.

Nonlinear least squares problems are usually solved by the various implementations of Levenberg–Marquardt [1]–[3], which differ in the selection of λ and D. Independent of the details of implementation, (5) and (7) have a fixed point when $g(\mu) = J^T r = 0$. Unfortunately, this can happen for local minima that correspond to poor estimates of the parameters, as opposed to the desired global minimum $\hat{\mu}$ in (1). Conventional methods become practically useless for many problems where the desired solution can be obtained only with a "high-quality" initial guess.

Several groups have worked to improve the performance of nonlinear least squares methods. For example, [4] considers further refinement of (7), while [5] develops methods suited to specific class of residuals. In [6], Bertsekas presents a method based on the extended Kalman Filter