Limitation of Markov Models and Event-Based Learning & Optimization

Plenary Presentation
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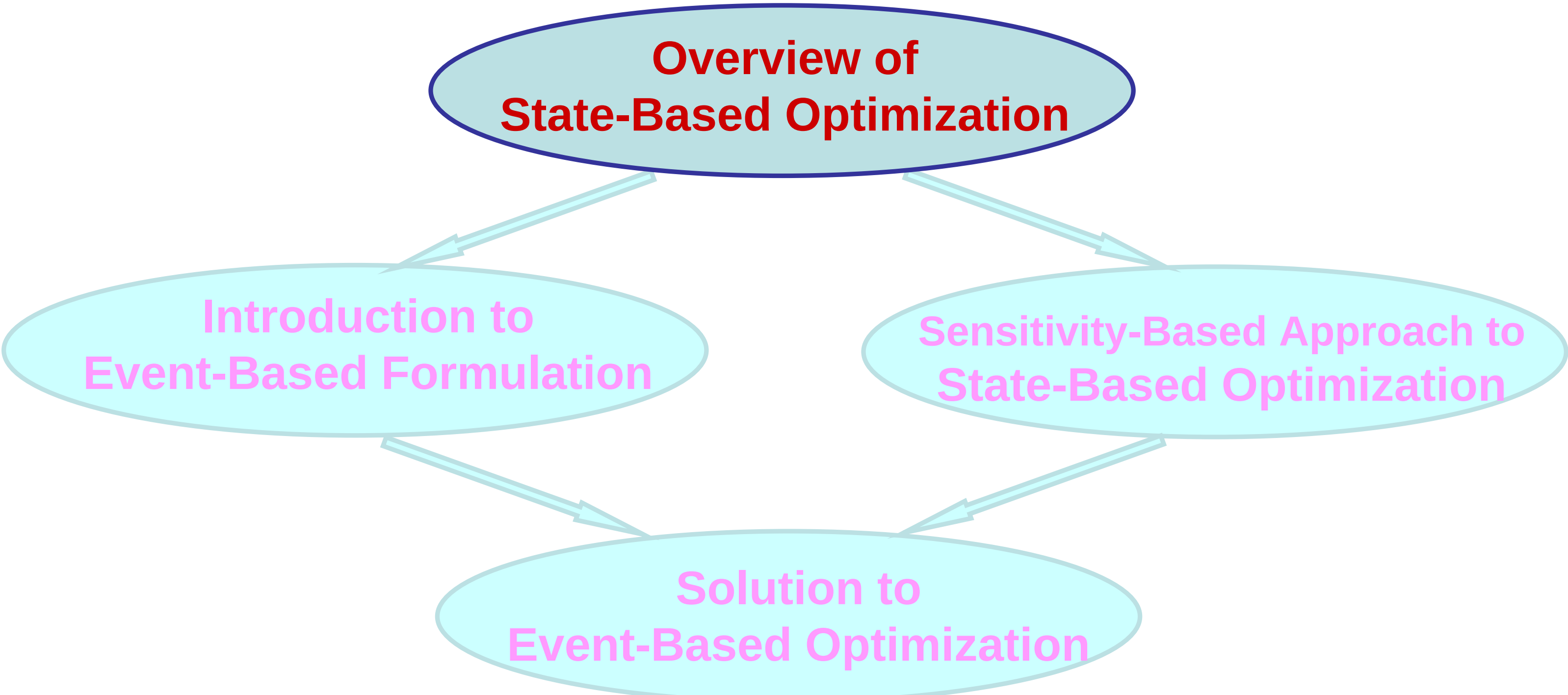
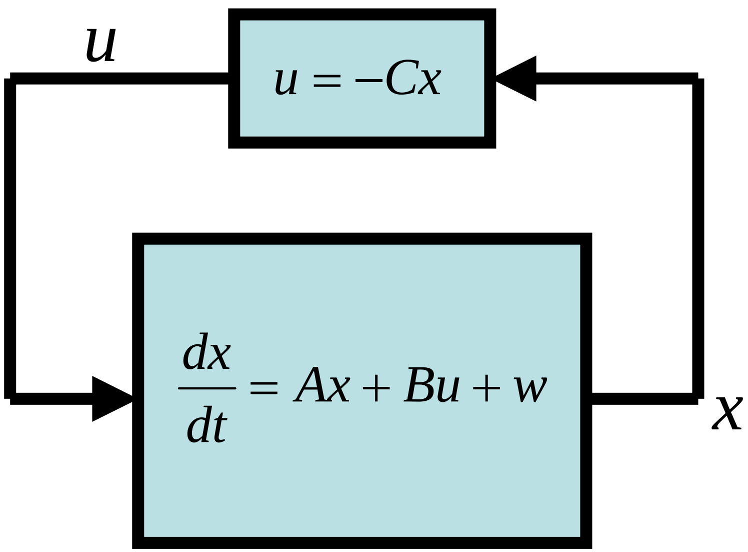

Overview of
State-Based Optimization
Introduction to
Event-Based Formulation
Sensitivity-Based Approach to
State-Based Optimization
Solution to
Event-Based Optimization

A Typical Formulation of a Control Problem
(Continuous Time Continuous State Model)



x: *State*

u: *Control variable*

w: *Random noise*

Control u depends on state x

A policy u(x): x →u

Performance measure

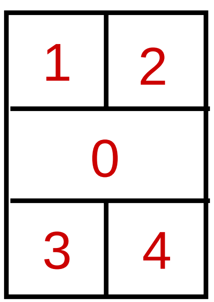
$$\eta = \frac{1}{T}\int_0^T E\{f[x(t), u(t)]\}dt$$

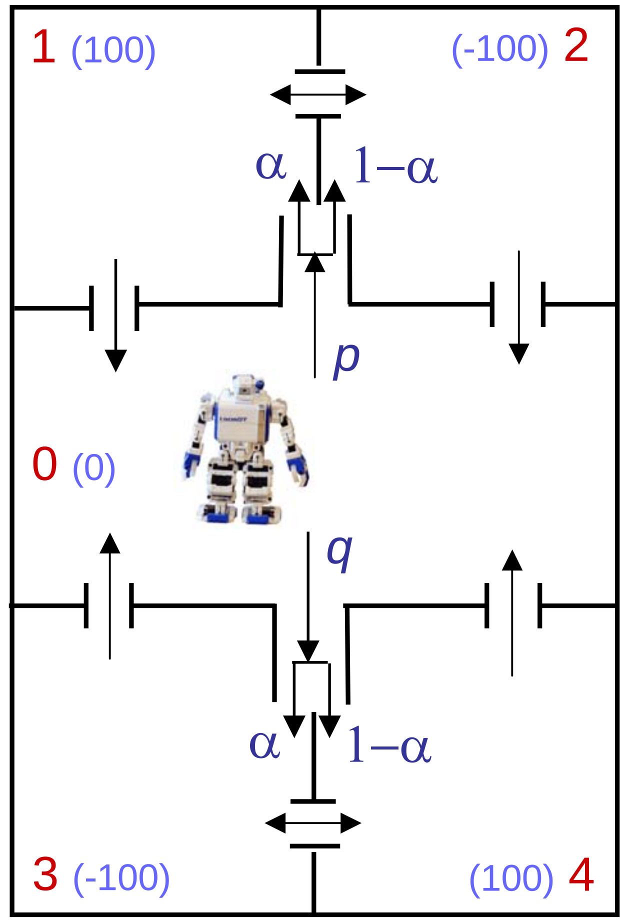
LQG problem

$$\eta = \frac{1}{T}\int_0^T E\{x^\tau Ax + u^\tau Bu\}dt$$

Discrete-time Discrete State Model (I)
- an example

A random walk
of a robot





Probabilities

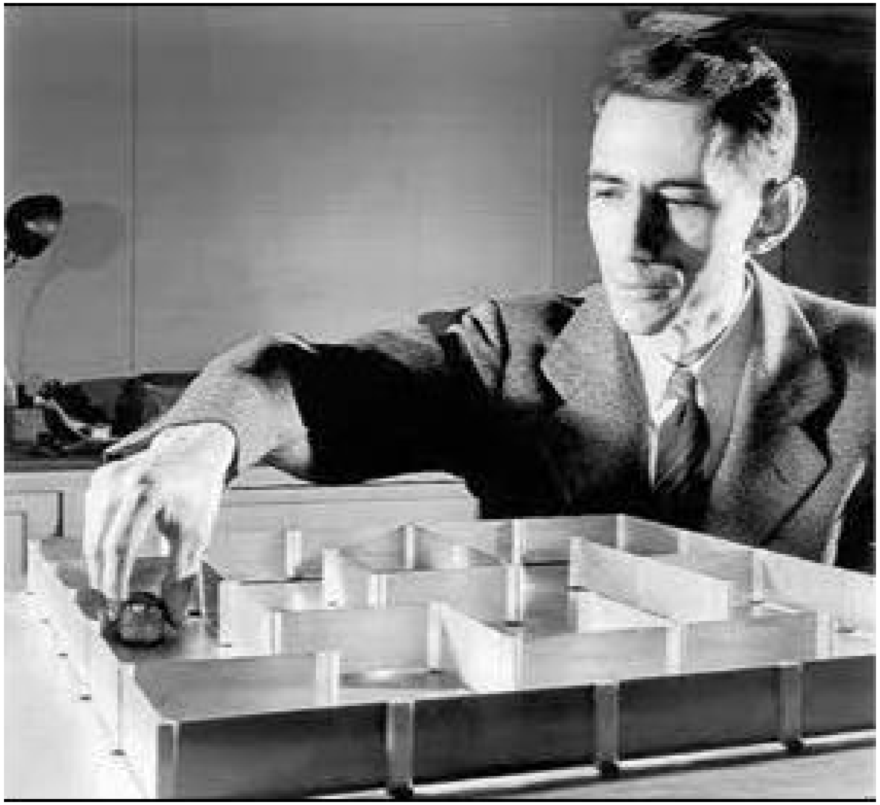
$$p + q = 1$$

Reward function

f(0) = 0
f(1) =f(4)=100
f(2) =f(3)= -100

Performance measure

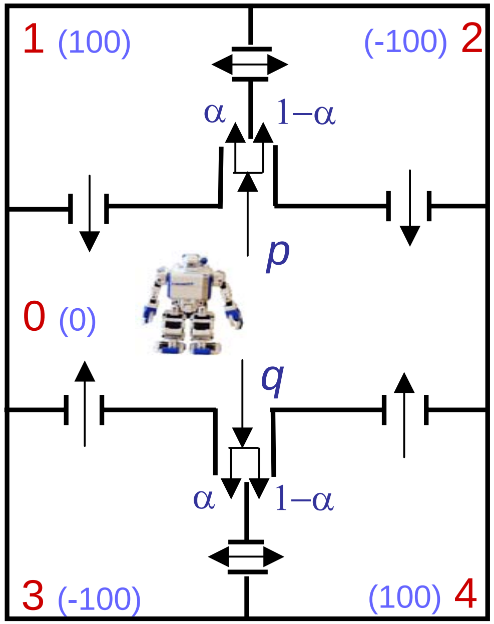
$$\eta = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(X_t)$$



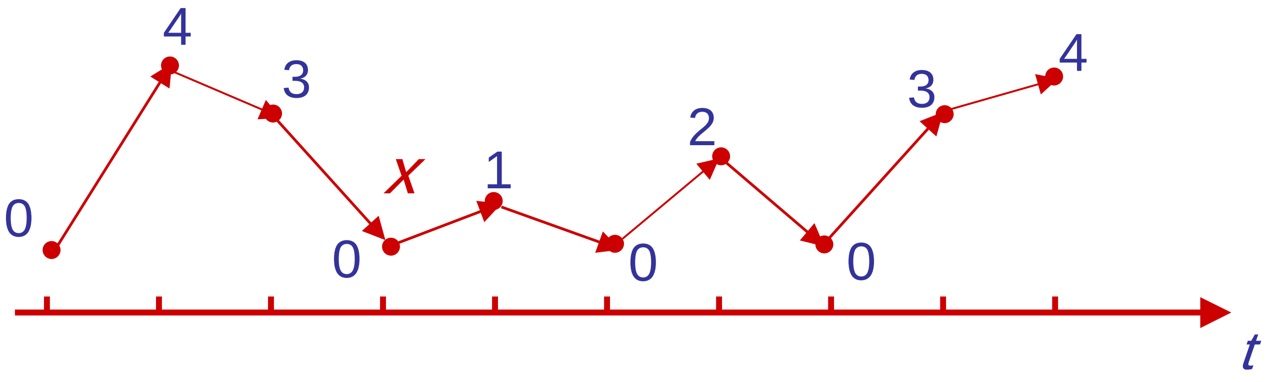
Shannon Mouse (Theseus)

Discrete Model (II)
- the dynamics

A random walk
of a robot

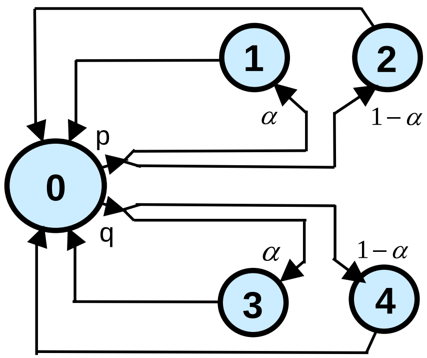


A Sample Path (system dynamics):



Discrete Model (III)
- the Markov model

Random Walker



System dynamics:

-X = {X_n, n=1,2,…}, X_n in S = {1,2,…,M}

- Transition Prob. Matrix $P=[p(i,j)]_{i,j=1,..,M}$

System performance:

– Reward function: $f=(f(1),\ldots,f(M))^T$

– Performance measure:

$$\eta = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(X_t) = \pi f = \sum_{i\in S} \pi(i) f(i)$$

Steady-state probability:

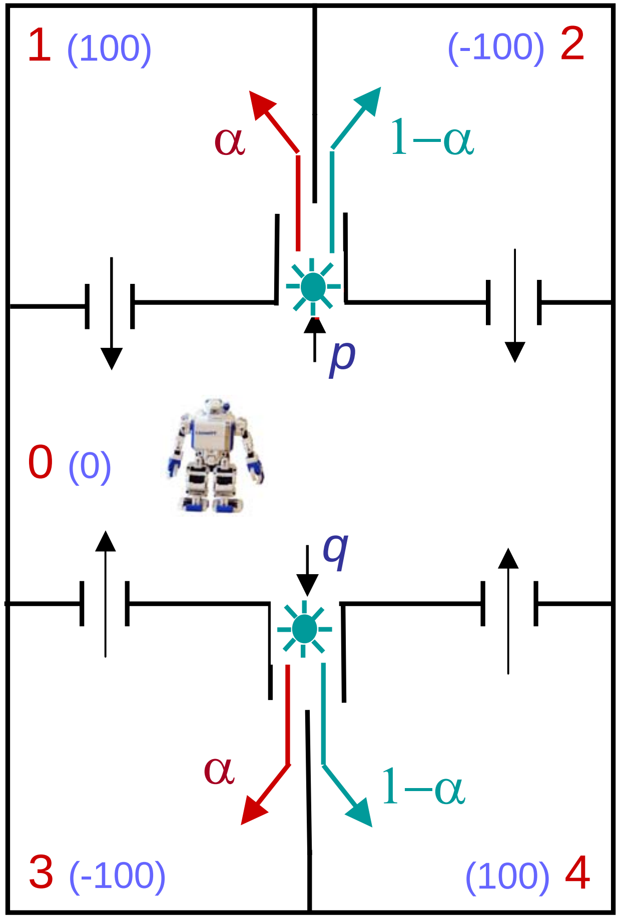
– Steady-state probability:

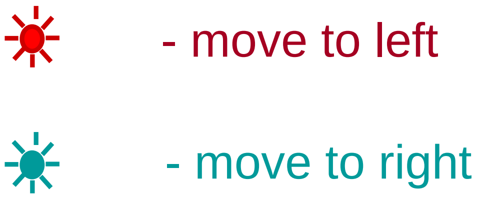
$\pi=(\pi(1), \pi(2),..,\pi(M))$.

$\pi(I-P)=0, \quad \pi e=1$

I:identity matrix, $e=(1,\ldots,1)^T$

Control of Transition Probabilities



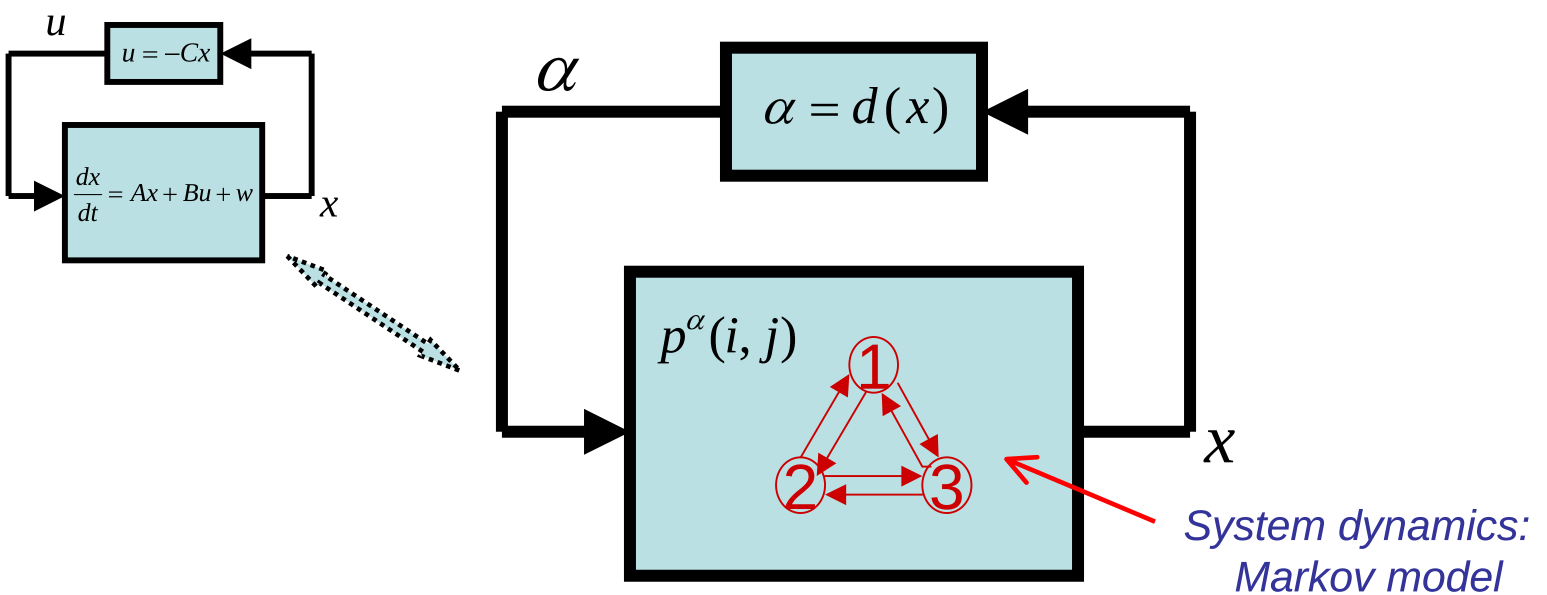


Turn on red with prob. α

Turn on green with prob. 1- α

Discrete Model (IV)
- Markov decision processes (MDPs)

- the Control Model



α: Action controls transition probabilities

$p^{\alpha}(i,j)$: governs the system dynamics

$\alpha=d(x)$: policy (state based)

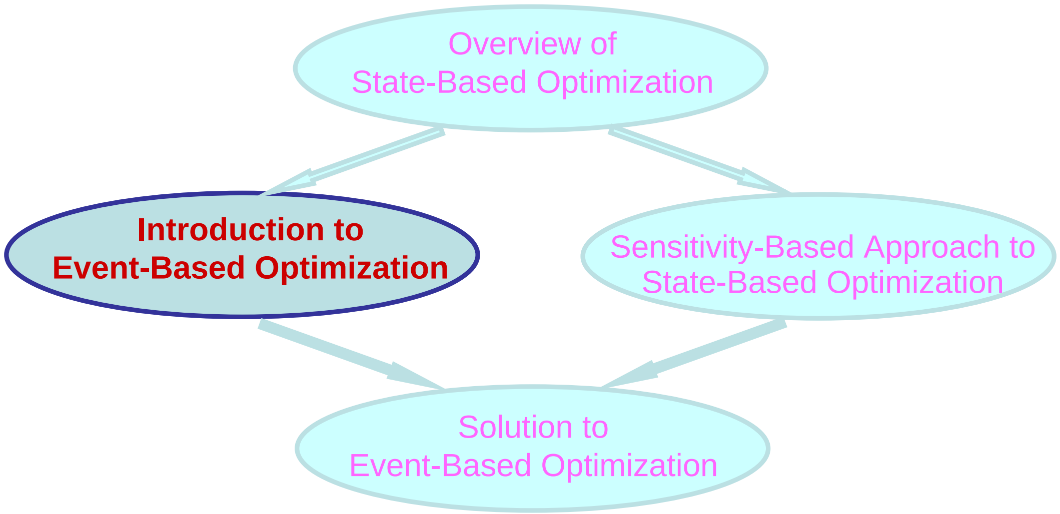
Performance depend on policies, π^d, η^d, etc

$$\eta^d = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} f(X_t^d)$$

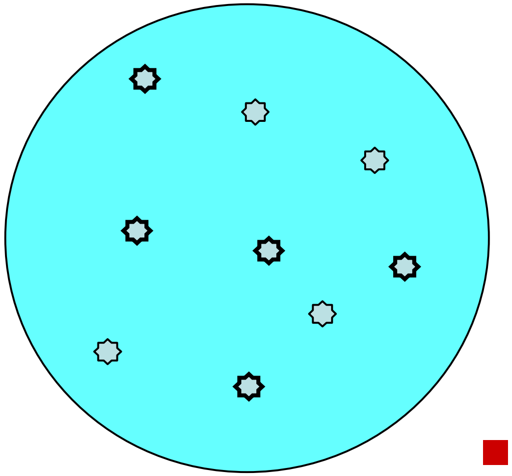
Goal of Optimization:
Find a policy d that maximizes η^d in policy space

0. Review: Optimization Problems (state-based policies)

1. Event-Based Optimization
   - Limitation of the state-based formulation
   - Events and event-based policies
   - Event-Based Optimization



Limitation of State-Based Formulation (I)



- The policy space is too large

  M = 100 states, N=2 actions,

  $N^M = 2^{100} = 10^{30}$ policies

  (10GHZ ➔ $3 * 10^{12}$ years to count!)

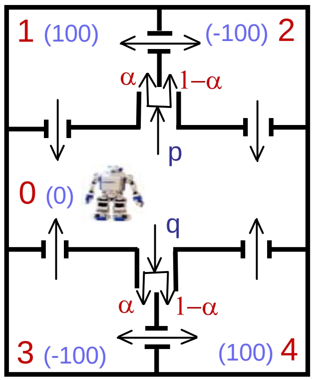
- Special structures not utilized

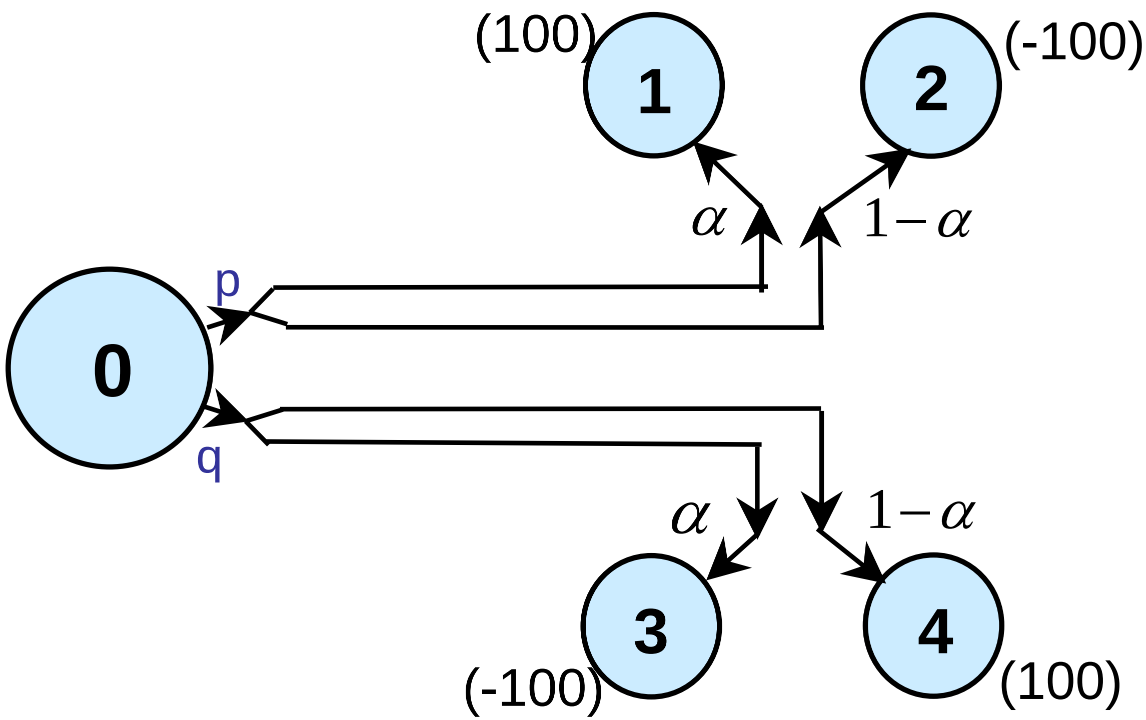
- May not perform well

Limitation of State-Based Formulation (II)

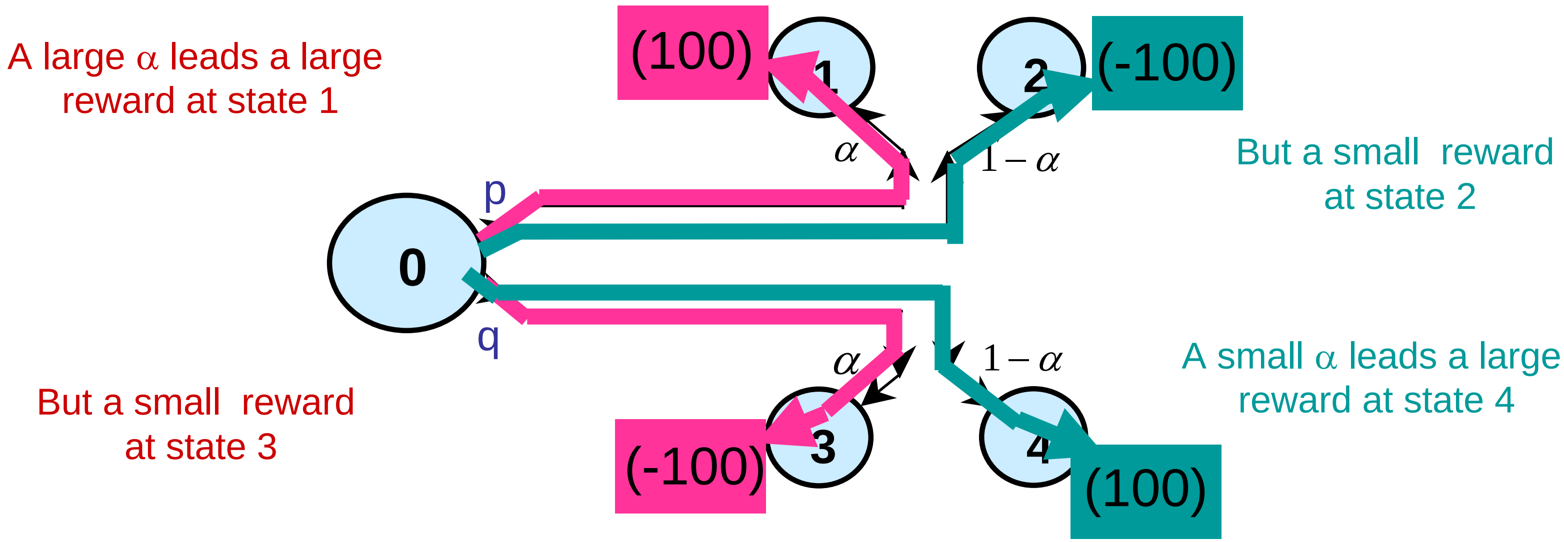
Example: Random walk of a robot

Choose α to maximize the average performance





Limitation of State-Based Formulation (III)

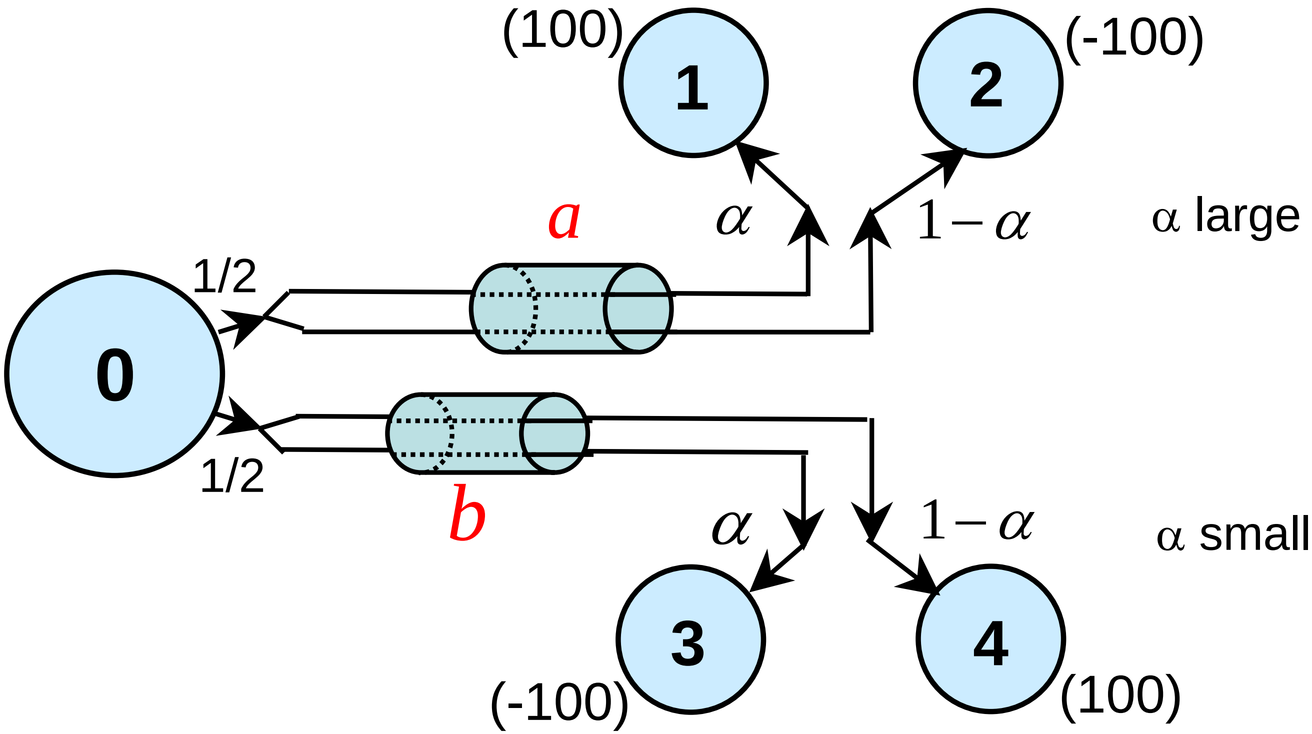


Transition probabilities:

	1	2	3	4
0	$p\alpha$	$p(1-\alpha)$	$q\alpha$	$q(1-\alpha)$

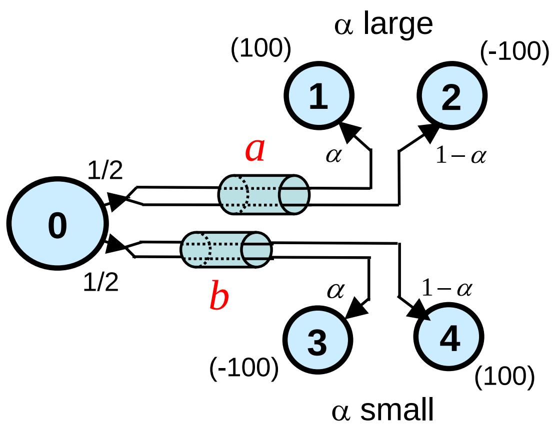
- At state 0,
  - ➔ if moves top, α needs to be as large as possible
  - ➔ if moves down, α needs to be as small as possible
- Let p = q = 1/2,
  - ➔ Average perf in next step = 0, no matter what α you choose (best you can do with a state-based model)

We can do better!



- Group two up transitions together as an event "a" and two down transitions as event "b".
- When "a" happens, choose the largest α, When "b" happens, choose the smallest α.
- Average performance $= 100$, if $\alpha=1$.

Events and Event-Based Policies



Event a *Event b*

	1	2	3	4
0	$p\alpha$	$p(1-\alpha)$	$q\alpha$	$q(1-\alpha)$

- An event is defined as a set of state transitions
- Event-based optimization:
  - May lead to a better performance than the state-based formulation
  - MDP model may not fit:
    - Only controls a part of transitions
    - An event may consist of transitions from many states
  - May reflect and utilize special structures
- Questions:
  - Why it may be better?
  - How general is the formulation?
  - How to solve event-based optimization problems?

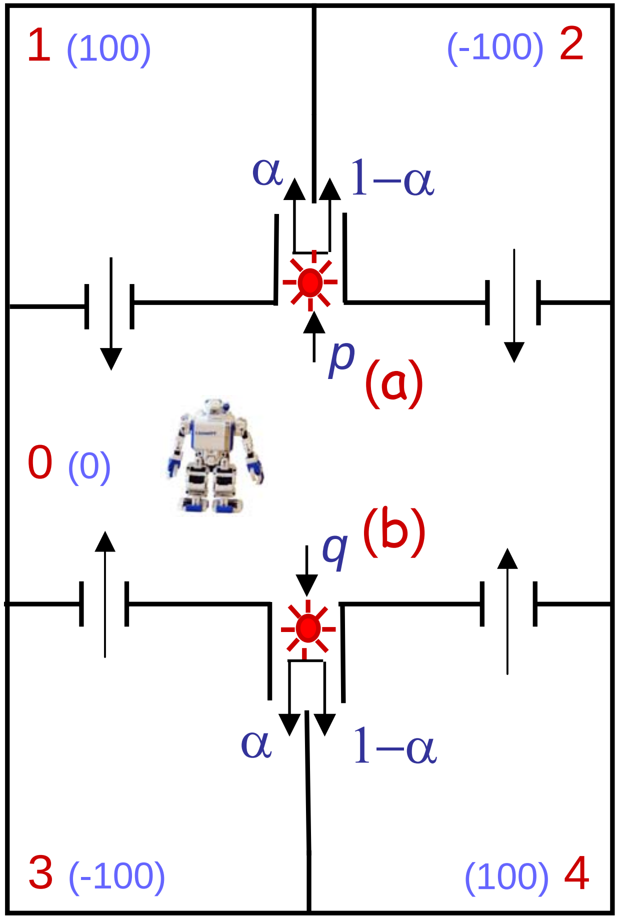
Notations:

- A single transition <i,j>,
  i,j in S ={1,2, …, M}
- An event: a set of transitions,
  2^M sets
  a = {<0,1>, <0,2>}
  b = {<0,3>, <0,4>}

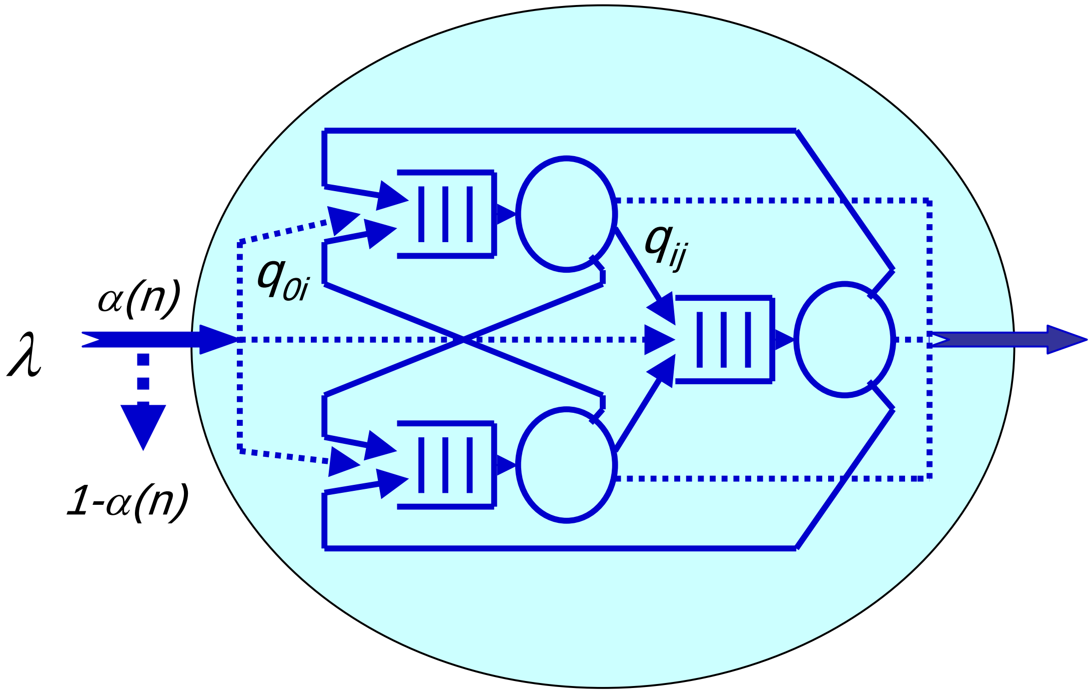
Why it is better?

An event contains information about the future!
(compared with the state-based policies)

Physical interpretation



How general is the formulation?



Admission control

n: population

No. of customers in network

n_i: No. of customers at server i

n$=(n_1,\ldots,n_M)$*: state*

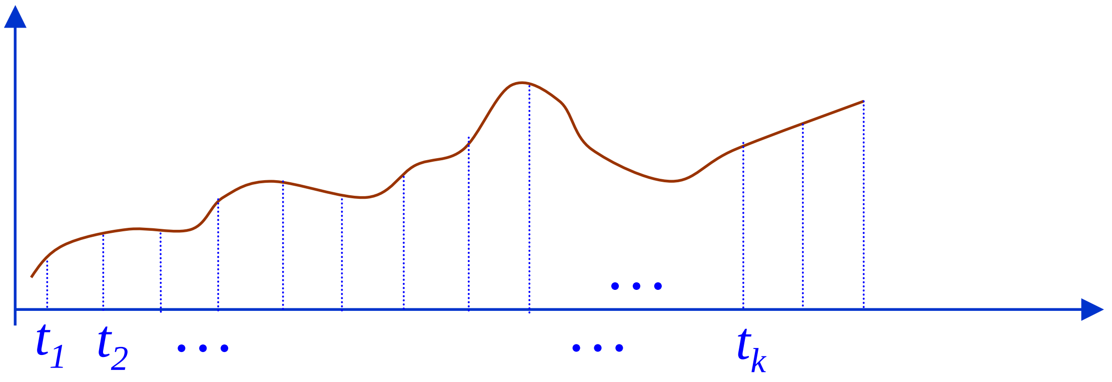
N: network capacity

- Event: a customer arrival finding population n
- Action: accept or reject

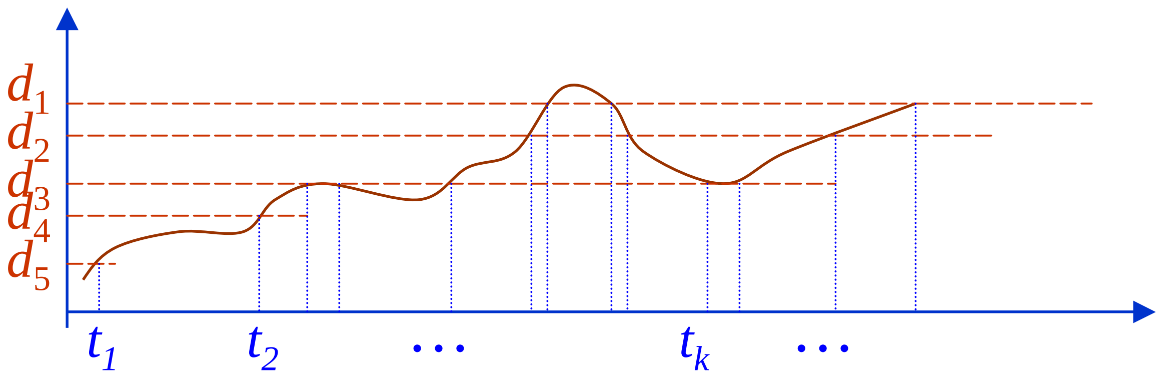
  Only applies when an event occurs
- MDP does not apply: Same action is applied for different state with the same population

Riemann Sampling vs. Lebesgue Sampling

RS:

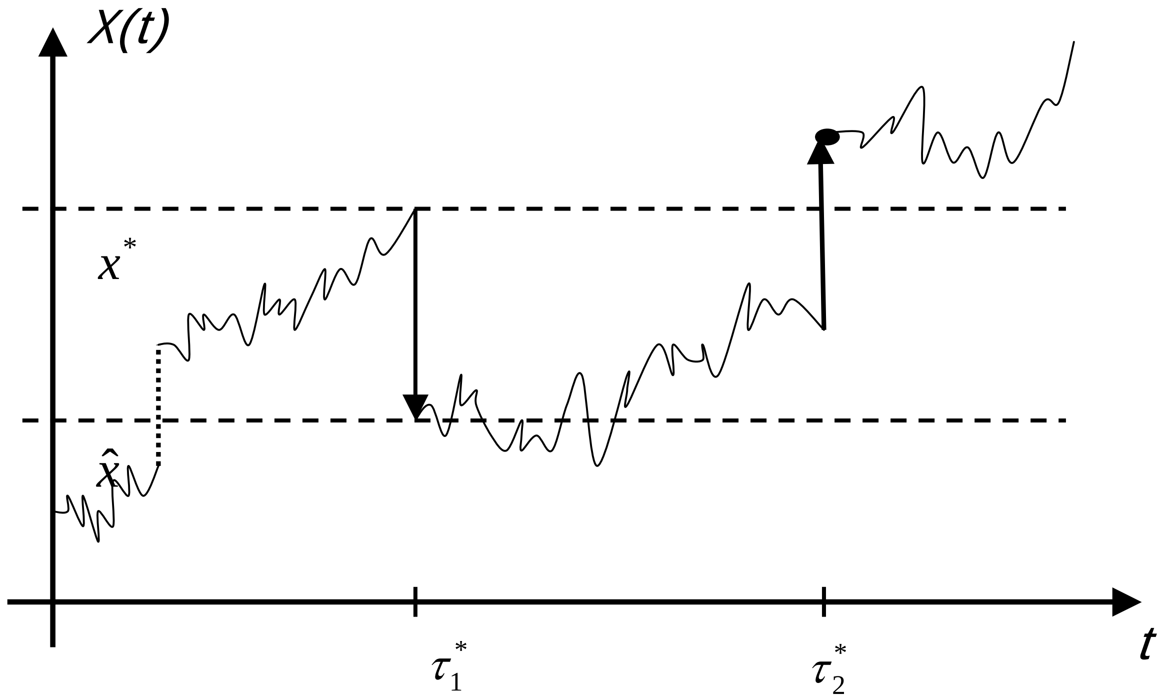


LS:



Sample the system whenever the signal reaches a certain prespecified level, and control is added then.

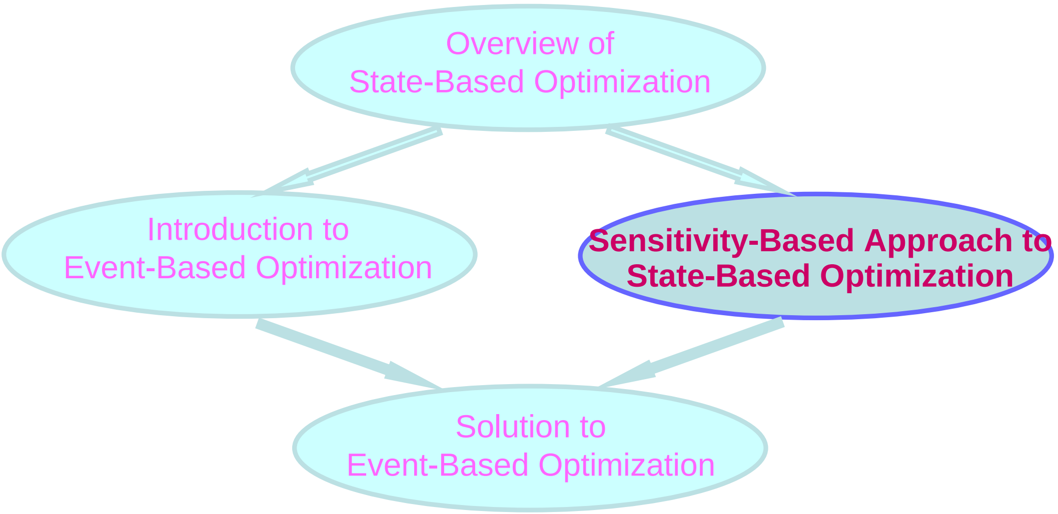
A Model for Stock Price or Financial Assess



$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dw(t) + \int \gamma(t, X(t-), z)N(dt, dz).$$

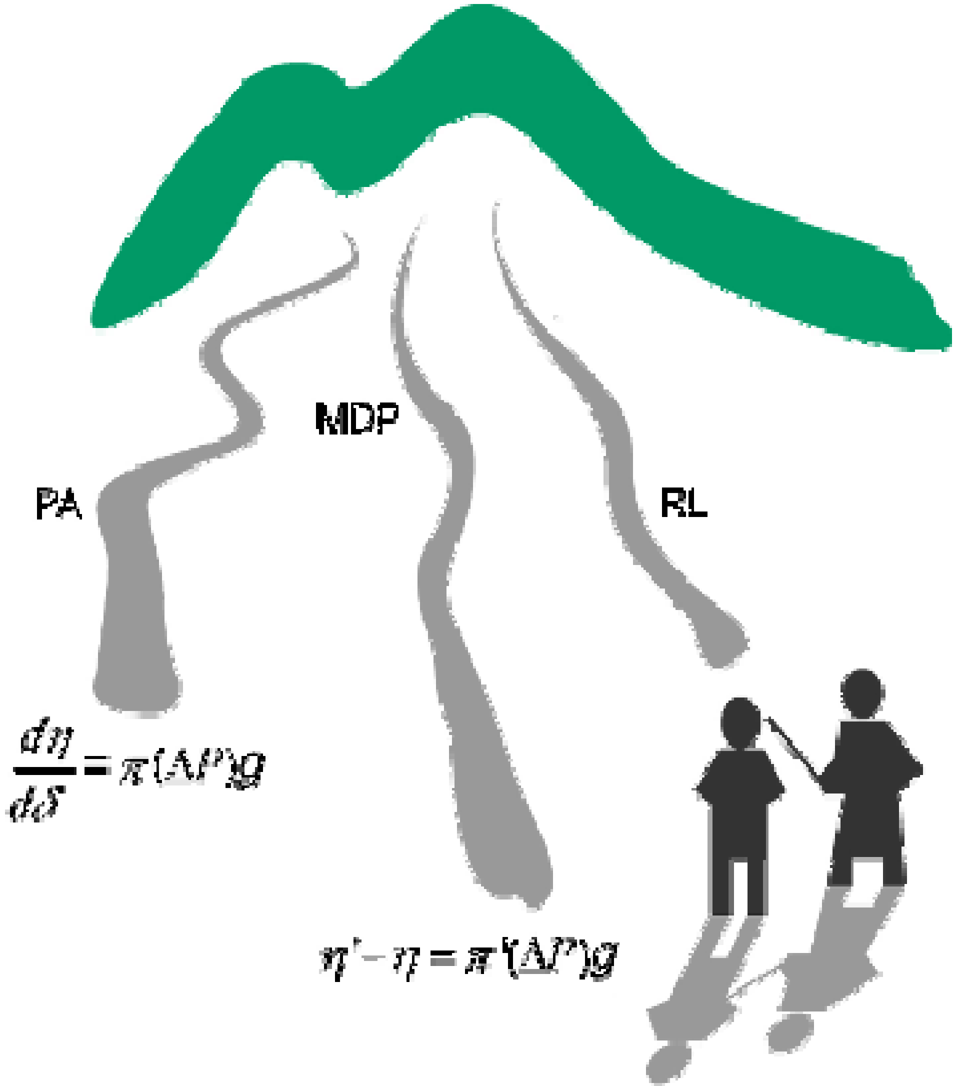
w(t): Brownian motion; N(dt,dz): Poisson random measure

X(t): Ito-Levy process



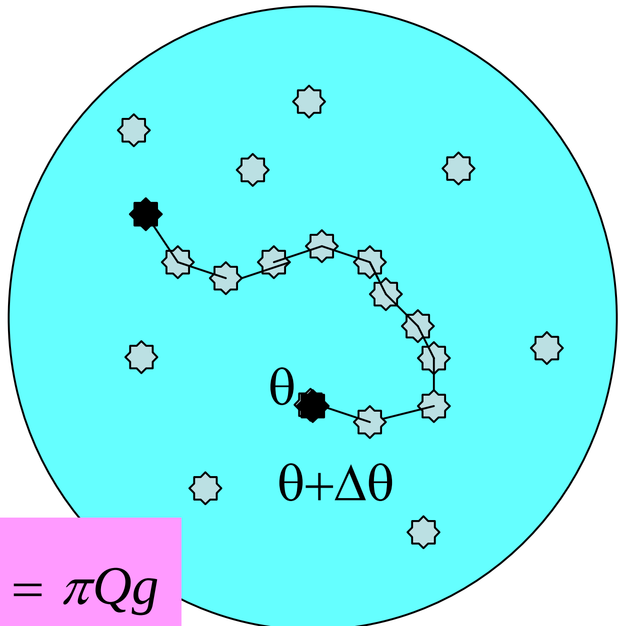
2. Sensitivity-Based Approach to Optimization
   - A unified framework for optimization
   - Extensions to event-based optimization
3. Summary

An overview of the paths to the top of a hill

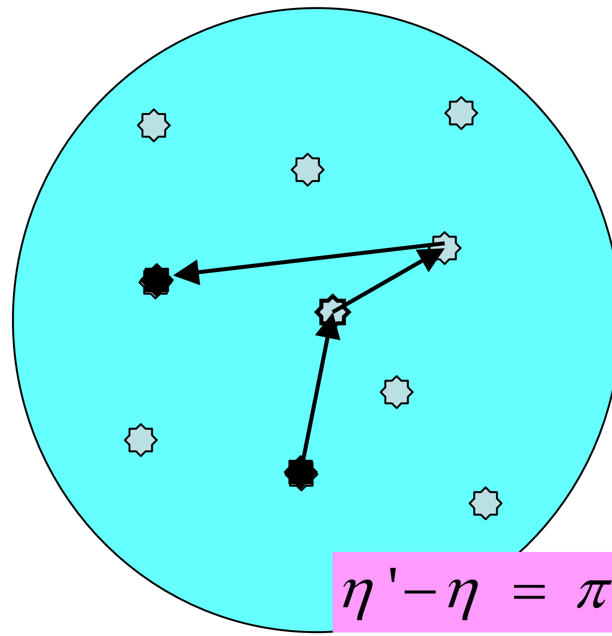


A Sensitivity-Based View of Optimization

- Continuous Parameters (perturbation analysis)
- Discrete Policy Space (policy iteration)



$$\frac{d\eta}{d\delta} = \pi Q g$$



$$\eta' - \eta = \pi' Q g$$

η: *performance*
π: *steady-state prob*
g: perf. potentials
Q=P'-P

Poisson Equation

g(i) = potential contribution of state i (potential, or bias)
= contribution of the current state f(i)-η
+ expected long term contribution after a transition

$$g(i) = f(i) - \eta + \sum_{j=1}^{M} p(i,j) g(j)$$

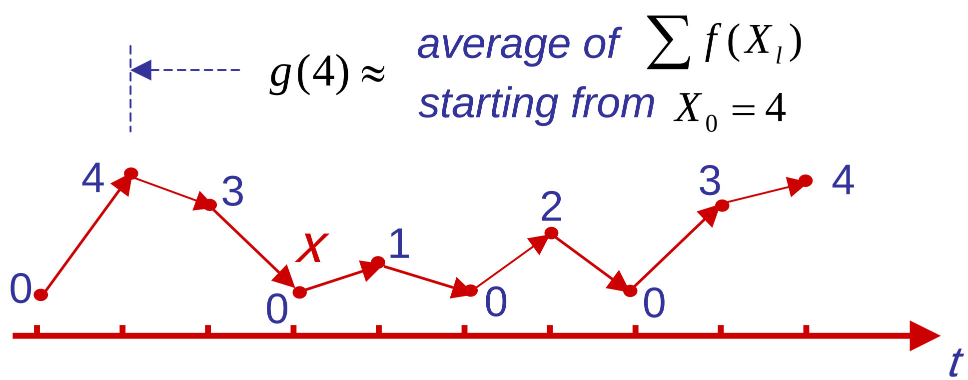
In matrix (Poisson equation):

$$(I - P)g + \eta e = f$$

Potential is relative: if g(i) is solution, i=1,..., M, so is g(i)+c, c: constant

Physical interpretation:

$$g(i) = E\{\sum_{l=0}^{\infty}[f(X_l) - \eta] | X_0 = i\}$$



Two Sensitivity Formulas

For two Markov chains *P*, η, π and *P'*, η', π', let *Q=P'-P*

Performance difference:

$$\eta' - \eta = \pi' Q g = \pi'(P' - P) g$$

One line simple derivation: $\times \pi'$: $(I - P)g + \eta e = f$

Performance derivative:

P is a function of θ*: P(*θ*)*

$$\frac{d\eta(\theta)}{d\theta} = \pi \frac{dP(\theta)}{d\theta} g \quad = \quad \frac{d}{d\theta}[\pi P(\theta) g]$$

Derivative =average change in expected potential at next step

Perturbation analysis: choose the direction with the largest average change in expected potential at next step

Policy Iteration

$$\eta' - \eta = \pi' Q g = \pi' (P' - P) g$$

1. $\eta' > \eta$ *if* $P'g > Pg$ *(Fact:* $\pi' > 0$ *)*

2. *Policy iteration:*
   At any state find a policy P' with P'g>Pg

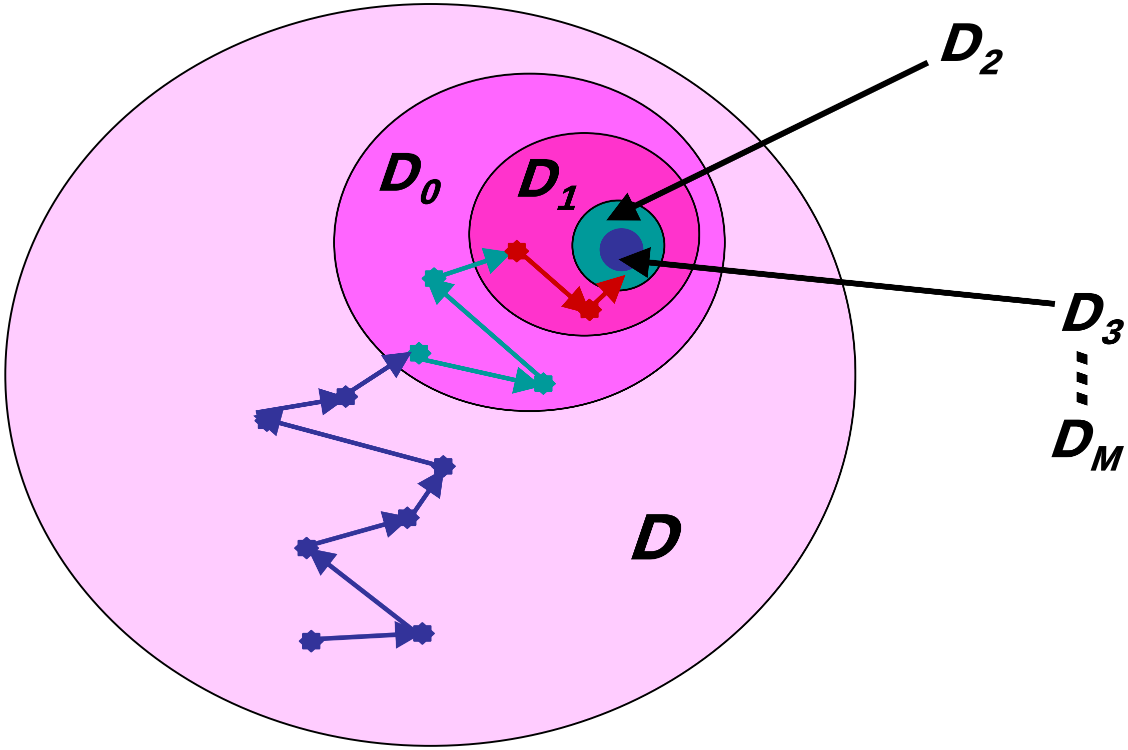
Policy iteration: Choose the action with largest changes in expected potential at next step

3. *Reinforcement learning*
   (Stochastic approximation algorithms)

Mutli-Chain MDPs

Perf./ Bias/ Blackwell Optimization

With perf. difference formulas, we can derive a simple, intuitive approach without discounting



***D*:** *Policy space*

***D_0*:** *Perf. optimal policies*

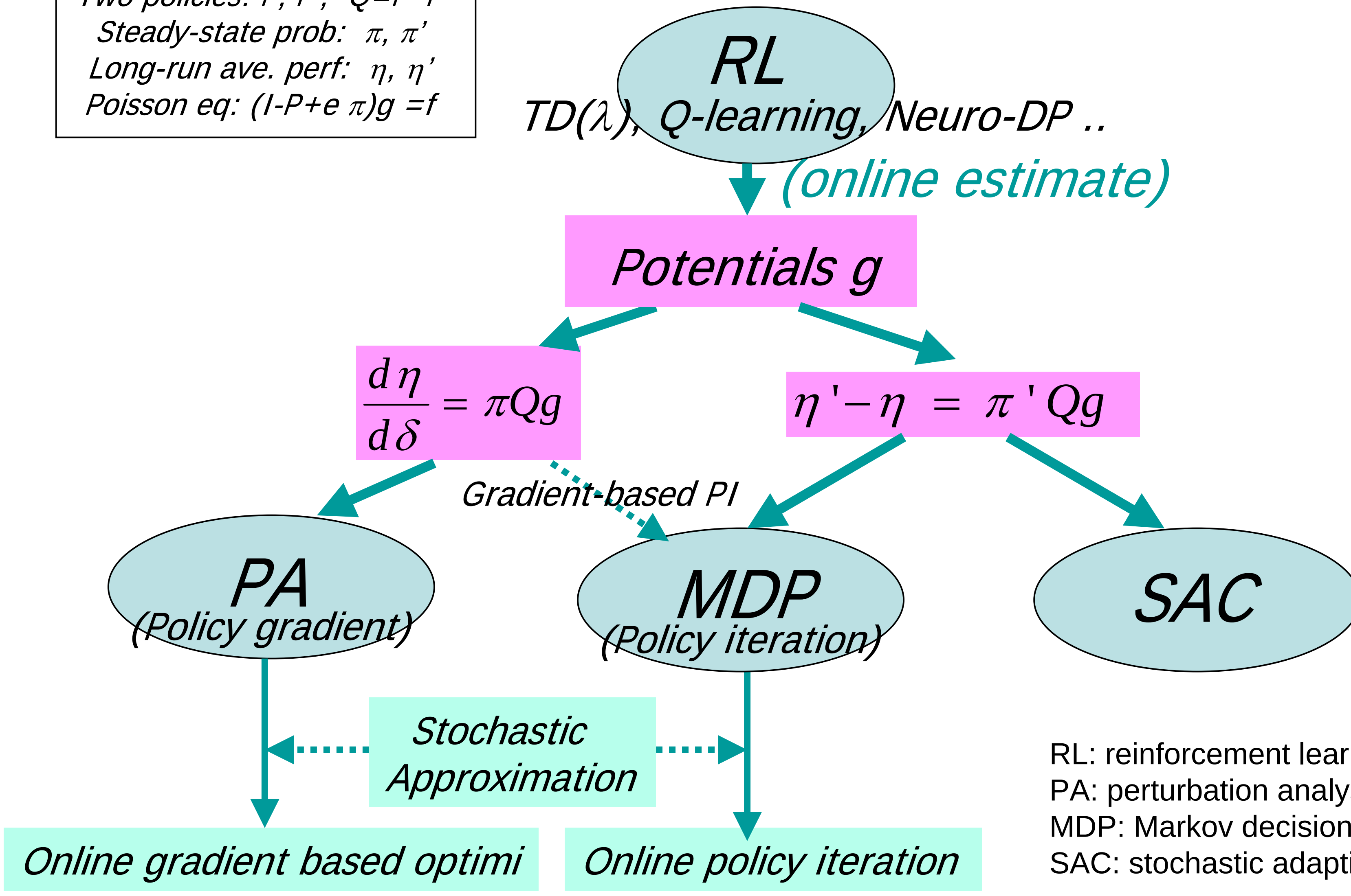
***D_1*:** *(1^{st}) Bias optimal policies*

***D_2*:** *2^{nd} Bias optimal policies*

…… ***D_M*:** *Blackwell optimal policies*

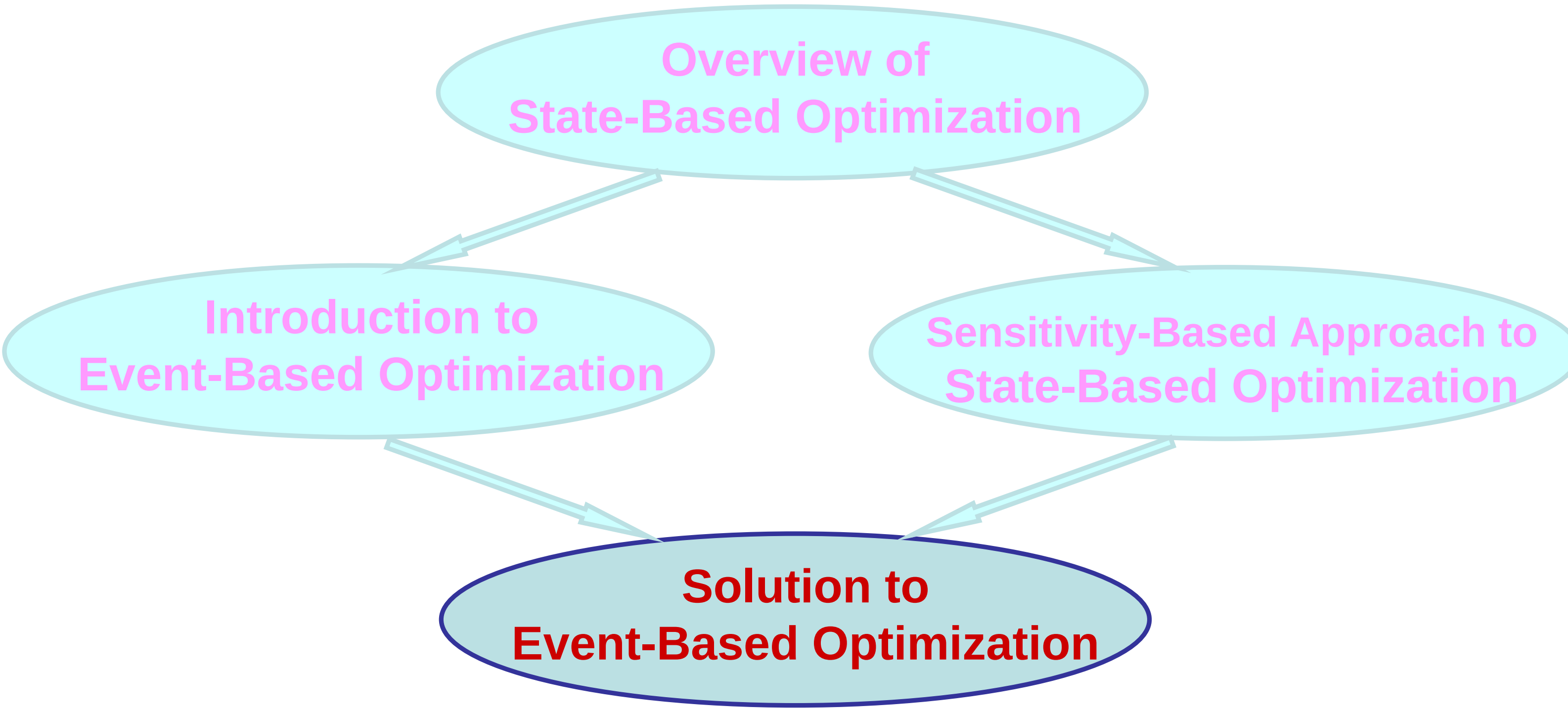
Bias measures transient behavior

Two policies: P, P', Q=P'-P
Steady-state prob: π, π'
Long-run ave. perf: η, η'
Poisson eq: $(I-P+e\,\pi)g = f$



RL: reinforcement learning
PA: perturbation analysis
MDP: Markov decision proc.
SAC: stochastic adaptive cont.

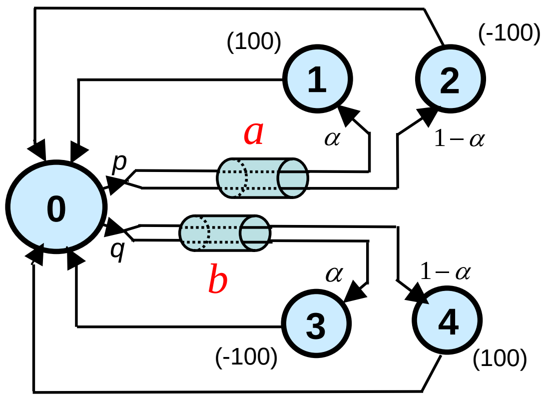
A Map of the L&O World



Extension of the sensitivity-based approach
to event-based optimization

- Two sensitivity formulas
  - Performance derivatives
  - Performance differences
- PA & PI
  - PA: Choose the direction with largest average change in expected potential at next step
  - PI: Choose the action with largest changes in expected potential at next step
- Potentials are aggregated according to event structure

Solution to
Random Walker Problem



Two policies:

$$\alpha_a = d(a), \quad \alpha_b = d(b)$$
$$\alpha'_a = d'(a), \quad \alpha'_b = d'(b)$$

1. *Performance diff:*

$$\eta' - \eta = \pi'(a)[(\alpha_a' - \alpha_a)g(a)] + \pi'(b)[(\alpha_b' - \alpha_b)g(b)]$$
$$g(a) = g(1) - g(2) \quad g(b) = g(3) - g(4)$$

$\pi'(a)$, $\pi'(b)$: perturbed steady-state prob. of events a and b

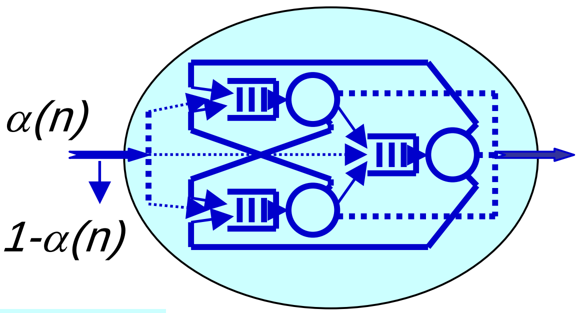
Choose the action with the largest changes
In expected potential at next step
g(a), g(b) aggregated

2. *Performance deriv:*

Continuous with θ: $\alpha_a(\theta)$, $\alpha_b(\theta)$

$$\frac{d\eta_\theta}{d\theta} = \pi_\theta(a)\frac{d\alpha_a(\theta)}{d\theta}[g_\theta(1) - g_\theta(2)] + \pi_\theta(b)\frac{d\alpha_b(\theta)}{d\theta}[g_\theta(3) - g_\theta(4)]$$

Solution to
Admission Control Problem



Two policies: $\alpha(n)$ and $\alpha'(n)$

1. *Performance diff:*

$$\eta' - \eta = \sum_{n=0}^{N-1} \{ p'(n) [\alpha'(n) - \alpha(n)] d(n) \}$$

$p(n)$: prob. of arrival finding n cust.

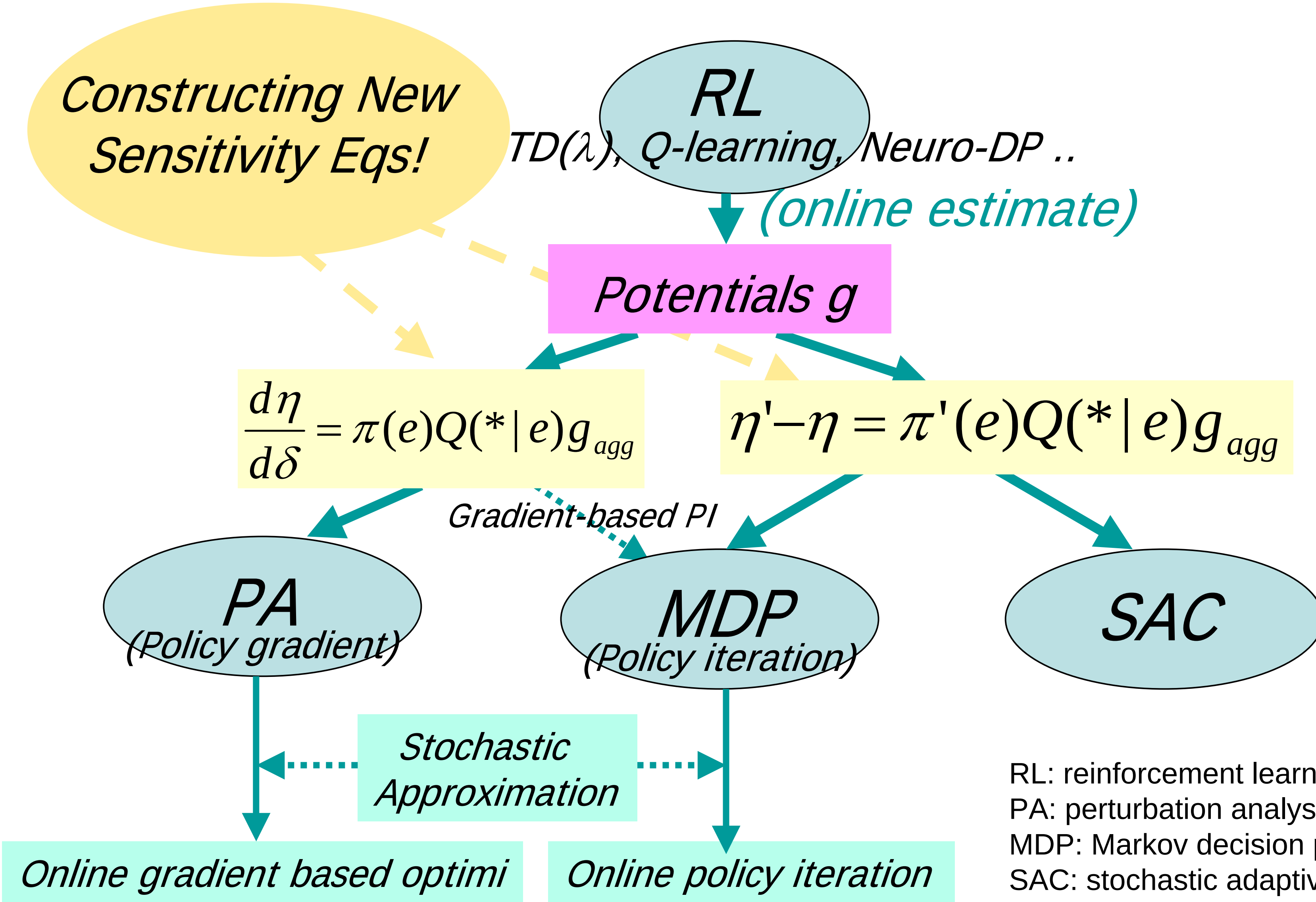
Potential aggregation:

$$d(n) = \frac{1}{p(n)} \{ \sum_{i=1}^{M} q_{0i} [\sum_{\sum n_i = n} p(\bar{n}) g(\bar{n}_{+i})] - \sum_{\sum n_i = n} p(\bar{n}) g(\bar{n}) \}$$

Choose the action with the largest changes
In expected potential at next step
$d(n)$: aggregated potential

2. *Performance deriv:*

$$\frac{d\eta}{d\delta} = \sum_{n=0}^{N-1} \{ p(n) [\alpha'(n) - \alpha(n)] d(n) \}$$



RL: reinforcement learning
PA: perturbation analysis
MDP: Markov decision proc.
SAC: stochastic adaptive cont.

Sensitivity-Based Approaches to Event-Based Optimization

Summary

Advantages of the Event-Based Approach

1. *May have better performance*

2. *# of aggregated potentials d(n): N*

   may be linear in system

3. *Actions at different states are correlated*

   standard MDPs do not apply

4. *Special features captured by events*

   action depends on future information

5. *Opens up a new direction*

   to many engineering problems

   POMDPs: observation y as event

   hierarchical control: mode change as event

   network of networks: transitions among subnets as events

   Lebesgue Sampling

Sensitivity-Based View of Optimization

1. *A map of the learning and optimization world:*

   Different approaches can be obtained from two sensitivity equations

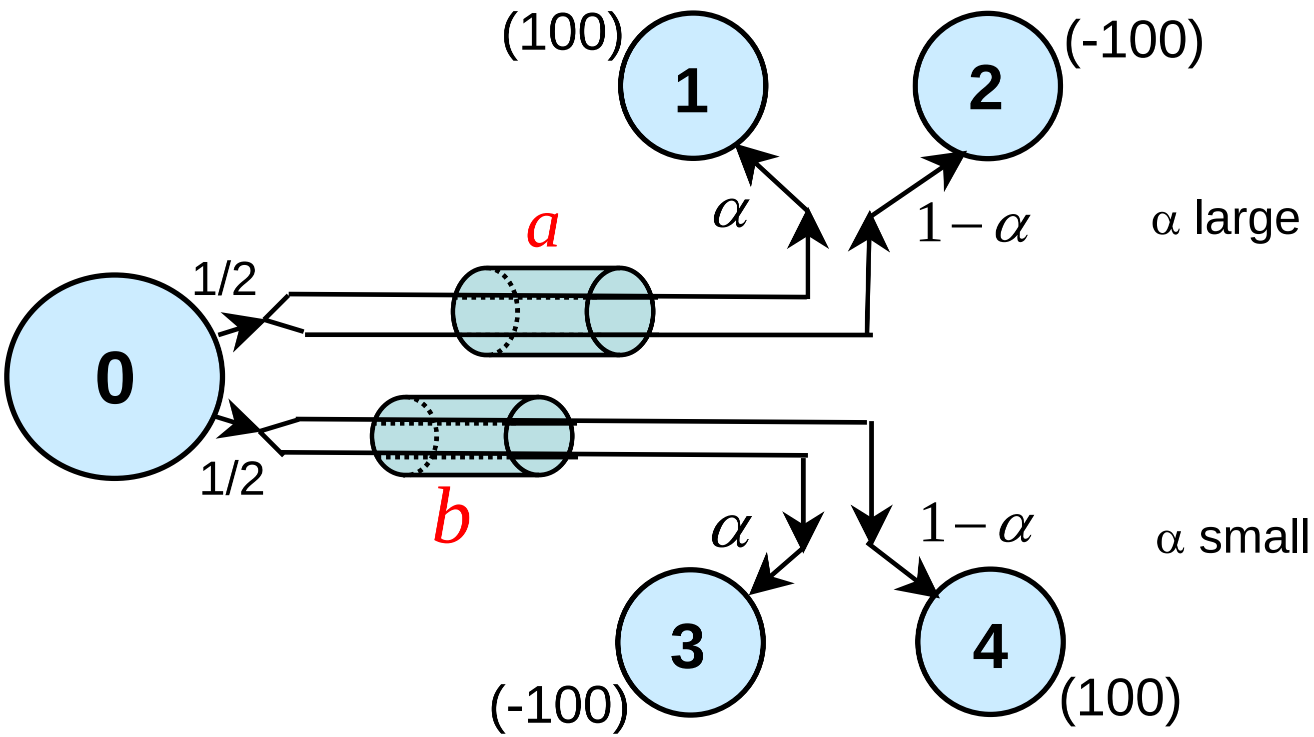
2. *Extension to event-based optimization*

   Policy iteration, perturbation analysis
   reinforcement learning, time aggregation
   stochastic approximation, Lebesgue sampling
   ……

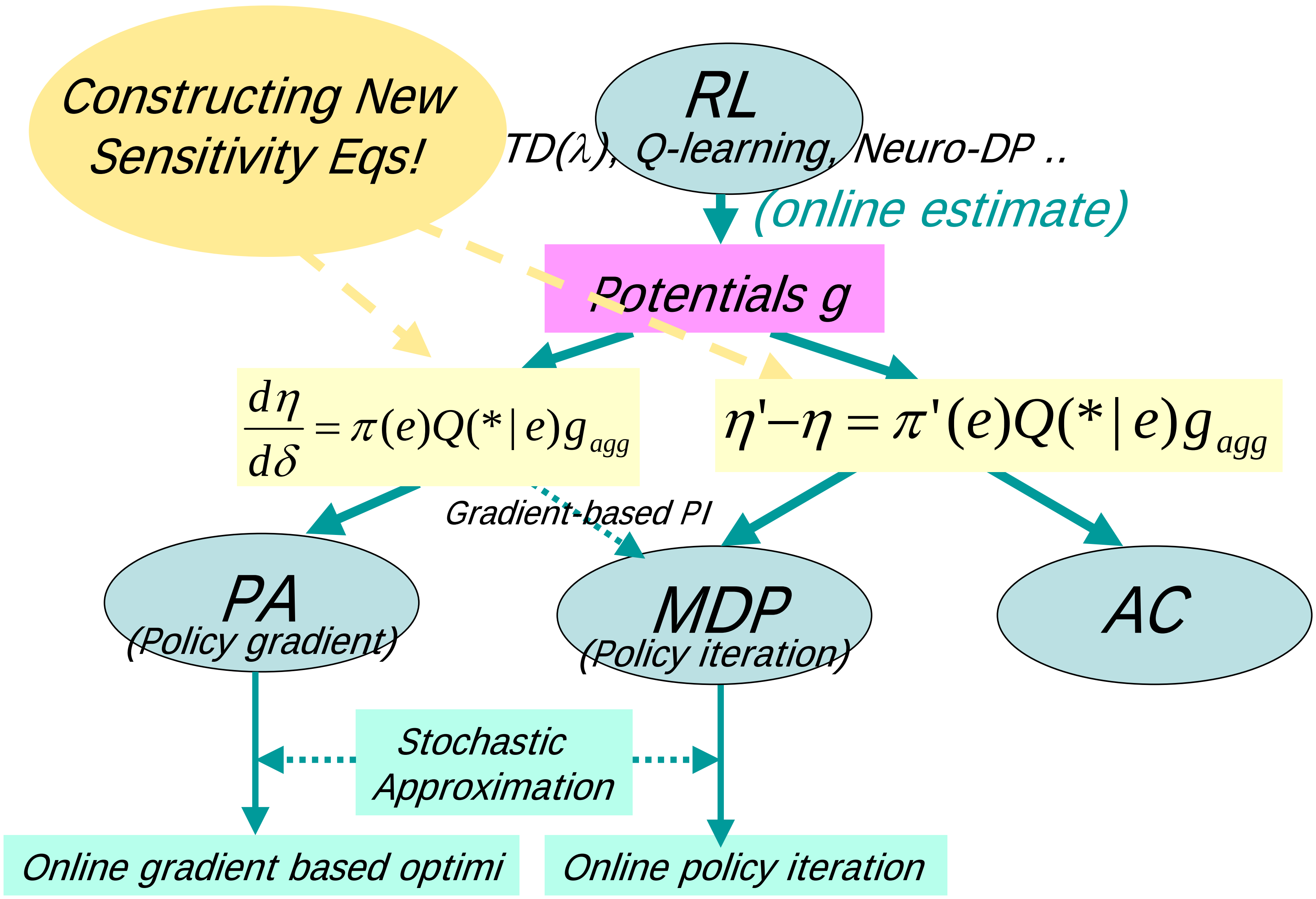
3. *Simpler and complete derivation for MDPs*

   Multi-chains, different perf. criteria
   Average performance with no discounting
   N-bias optimality – Blackwell optimality

Pictures to Remember (I)



Pictures to Remember (II)



Limitation of State-Based Formulation (I)

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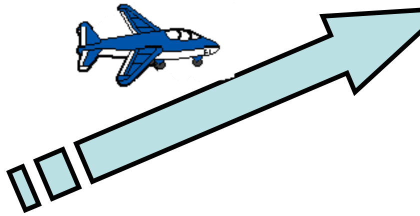
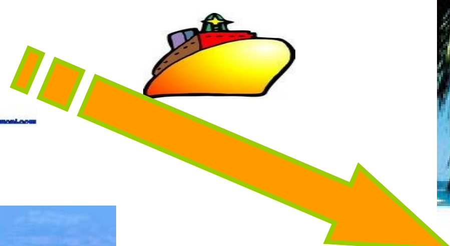




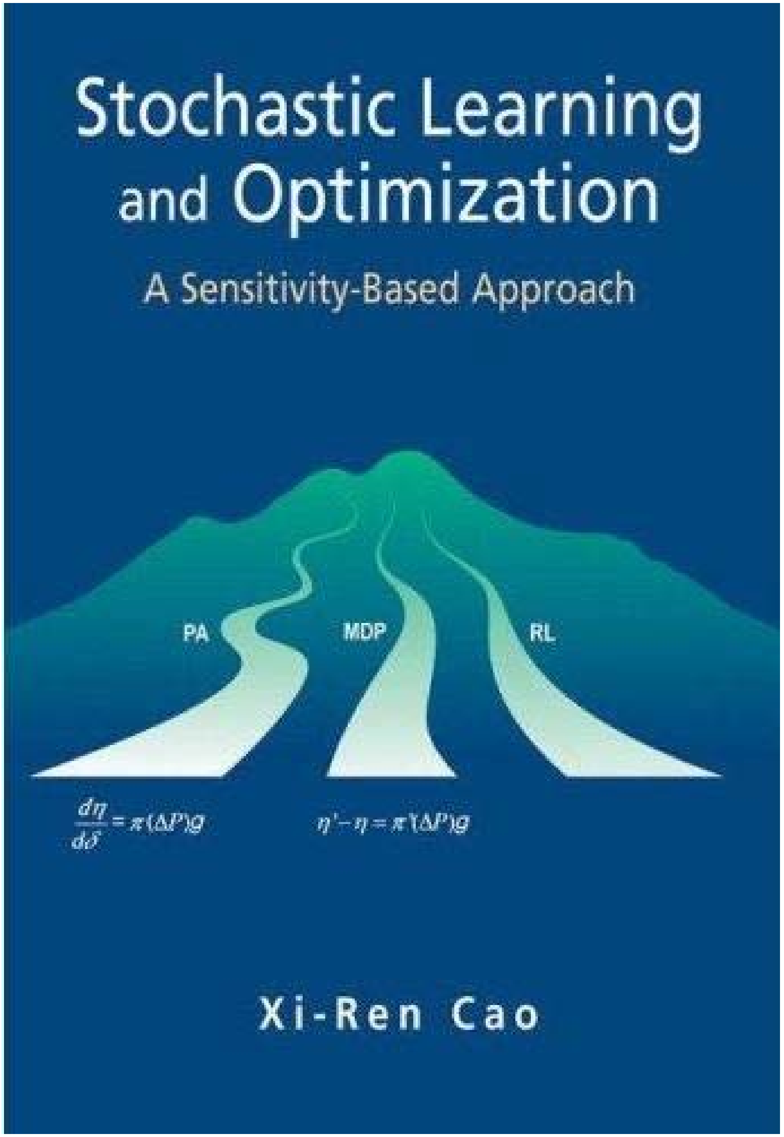
1 Alaska

2 Hawaii

0 Yautai

0	1
	2

Thank You!



Xi-Ren Cao:

Stochastic Learning and Optimization - A Sensitivity Based Approach

9 Chapters, 566 pages
119 Figures, 27 Tables,
212 homework problems

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